HOMEWORK 1 (TWO pages)
Due time: Before 13:00, Jan 26 (Thursday).

Each problem: 10 points.

For our first set of homework, I think the difficulty is on the level from medium to difficult. If you cannot finish all of them, it is perfectly fine. I intentionally designed this homework to measure the average level of the class. Difficulty of later homeworks will be adjusted. Again, I want to emphasize, please write down whatever you get, since partial credits will be given.

On your homework, please write down your full name.

(1) Suppose A is an Noetherian ring, I an ideal, and $B = A/I$ is the quotient ring of A. Show that:
 (a) B is a Noetherian A-module.
 (b) B is a Noetherian ring (i.e., B is a Noetherian B-module).

(2) Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is NOT a Noetherian $\mathbb{Z}$-module.

(3) Let k be a field, then $k[x]$ is an Euclidean domain.

(4) k a field, then the power series ring $k[[x]]$ is an Euclidean domain.

(5) Let $\mathbb{Z}[\sqrt{2}]$ be the set of elements $a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$. Show that it is a ring. Show that it is an Euclidean domain.

(6) $\mathbb{Z}[\sqrt{-2}]$ is an Euclidean domain.

(7) Let A be a PID. Let $a, b \in A$. If $x \in A$ such that x divides both a, b, then x is called a common divisor of a and b. Show that if the ideal (a, b) is equal to (d), then d is a *maximal* common divisor of a and b. Namely, if x is a common divisor of a and b, then x divides d.

(8) Let $M = \mathbb{Z}e \oplus \mathbb{Z}f$. Let M' be the submodule generated by $2e$ and $3f$. Find a basis of M' as in Part(2) of the Main theorem of "finite free modules over PID". (our Theorem 1.6). You need to show calculations. (not just final answers)

(9) Let $M = \mathbb{Z}e \oplus \mathbb{Z}f$. Let M' be the submodule generated by $5e + 2f$ and $4f$. Find a basis of M' as in Part(2) of Theorem 1.6. You need to show calculations.

(10) Let $A = \mathbb{Z}[i]$. Let $M = Ae \oplus Af$. Let M' be the submodule generated by $2e$ and $(3+i)f$. Find a basis of M' as in Part(2) of Theorem 1.6. You need to show calculations.

Addendum. The following are NOT exercises, they DON'T worth any points. They are more like suggested readings. You can either work on them by yourself, or you can find proofs in literature. If you write proofs and submit to me, I might look at them (if I have time).

(1) (Hilbert Basis Theorem) If A is Noetherian ring, then $A[x]$ is also Noetherian ring.

(2) If A is Noetherian ring, and B is a subring of A. Then B is NOT necessarily Noetherian.

(3) In the domain $\mathbb{Z}[\sqrt{-5}]$, there is an ideal generated by 3 and $1+\sqrt{-5}$. This ideal is NOT principal. (and so $\mathbb{Z}[\sqrt{-5}]$ is NOT PID).

Indeed, the fact that $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{-1}]$, $\mathbb{Z}[\sqrt{-2}]$ are PID's says that the number fields $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ "have class number 1". Look at wikipedia page named "List of number fields with class number one". There are many long-standing and difficult questions there.