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Solutions to the exercise set 6 (6 pages)

Exercise 2.5.7. The *trace* of a matrix A is defined as

$$\operatorname{tr} A := \sum_{i=1}^{n} (Ae_i \,|\, e_i),$$

where $(e_i)_{i=1}^n$ is any orthonormal basis. Show that this is well-defined, i.e. the result is independent of the chosen orthonormal basis. If G is a matrix-valued function, check that $\int_S \operatorname{tr} G(x)\,\mathrm{d}x = \operatorname{tr} \int_S G(x)\mathrm{d}x$.

Solution. Suppose that $(e_i)_{i=1}^n$ and $(f_i)_{i=1}^n$ are two orthonormal bases of $\mathbb{R}^n$. We simply notice that for every $i, j = 1, 2, \ldots, n$ we have

$$Ae_i = \sum_{j=1}^{n} (Ae_i \,|\, f_j) f_j \quad \Longrightarrow \quad (Ae_i \,|\, e_i) = \sum_{j=1}^{n} (Ae_i \,|\, f_j)(f_j \,|\, e_i)$$

and

$$f_j = \sum_{i=1}^{n} (f_j \,|\, e_i) e_i \quad \Longrightarrow \quad Af_j = \sum_{i=1}^{n} (f_j \,|\, e_i) Ae_i \quad \Longrightarrow \quad (Af_j \,|\, f_j) = \sum_{i=1}^{n} (f_j \,|\, e_i)(Ae_i \,|\, f_j).$$

Thus, we get

$$\sum_{i=1}^{n} (Ae_i \,|\, e_i) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} (Ae_i \,|\, f_j)(f_j \,|\, e_i) \;=\; \sum_{j=1}^{n} \sum_{i=1}^{n} (Ae_i \,|\, f_j)(f_j \,|\, e_i) \;=\; \sum_{j=1}^{n} (Af_j \,|\, f_j).$$

Since we just showed that the definition of the trace is independent of the chosen orthonormal basis, we may prove the latter claim for the standard orthonormal basis of $\mathbb{R}^n$, i.e. we simply set $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$. For this basis, we get the "classical" definition of the trace: if $A = [a_{ij}]_{i,j=1}^n$, then

$$Ae_i = \sum_{j=1}^{n} a_{ji} e_j \quad \Longrightarrow \quad (Ae_i \,|\, e_i) = a_{ii} \quad \Longrightarrow \quad \operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

Now, since $\int_S (G(x))_{ij}\,\mathrm{d}x = \left(\int_S G(x)\,\mathrm{d}x\right)_{ij}$ for every i and j, we get

$$\int_S \operatorname{tr} G(x)\,\mathrm{d}x \;=\; \int_S \sum_{i=1}^{n} (G(x))_{ii}\,\mathrm{d}x \;=\; \sum_{i=1}^{n} \int_S (G(x))_{ii}\,\mathrm{d}x \;=\; \sum_{i=1}^{n} \left(\int_S G(x)\,\mathrm{d}x \right)_{ii} \;=\; \operatorname{tr} \int_S G(x)\,\mathrm{d}x.$$

□

Exercise 2.5.8. Let A be a positive self-adjoint matrix. Show that $\|A\|_{\mathrm{op}} \leq \operatorname{tr} A \leq n\|A\|_{\mathrm{op}}$.

Solution. Suppose that $(e_i)_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$ and that $x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^n$ with $\|x\| = 1$. Since A is a positive self-adjoint matrix, we have $A = \sum_{i=1}^n \lambda_i e_i \otimes e_i$, where $e_i \otimes e_i = e_i(e_i \,|\;)$ and $\lambda_i > 0$. For the trace, we get

$$\operatorname{tr} A \;=\; \sum_{i=1}^n (Ae_i \,|\, e_i) \;=\; \sum_{i=1}^n \sum_{j=1}^n \lambda_j (e_j \otimes e_j(e_i) \,|\, e_i) \;=\; \sum_{i=1}^n \lambda_i.$$

For the operator norm, we get

$$\begin{aligned} \|Ax\|^2 \;=\; (Ax \,|\, Ax) \;=\; \left(\sum_{i=1}^n \lambda_i (e_i \,|\, x) \;\middle|\; \sum_{j=1}^n \lambda_j e_j (e_j \,|\, x) \right) \;&=\; \sum_{i=1}^n \lambda_i (e_i \,|\, x) \sum_{j=1}^n \lambda_j (e_j \,|\, x)(e_i \,|\, e_j) \\ &=\; \sum_{i=1}^n \lambda_i^2 (e_i \,|\, x)^2 \\ &\leq\; \max_j \lambda_j^2 \cdot \sum_{i=1}^n x_i^2 \\ &=\; \max_j \lambda_j^2 \cdot \|x\|. \end{aligned}$$

On the other hand, since $Ae_j = \sum_{i=1}^n \lambda_i e_i \otimes e_i(e_j) = \lambda_j e_j$, we have $\|Ae_j\| = \lambda_j$ for every j and thus, $\|A\|_{\mathrm{op}} \geq \max_j \lambda_j$. Hence, we have $\operatorname{tr} A = \sum_{i=1}^n \lambda_i$ and $\|A\|_{\mathrm{op}} = \max_j \lambda_j$. In particular,

$$\|A\|_{\mathrm{op}} \;=\; \max_j \lambda_j \;\leq\; \sum_{j=1}^n \lambda_j \;=\; \operatorname{tr} A \;\leq\; n \cdot \max_j \lambda_j \;= n\|A\|_{\mathrm{op}}.$$

□

Exercise 2.5.9. Prove that

$$\fint_S \|W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}^2 \mathrm{d}x \le c_n.$$

Solution. We use the facts $\|A\|_{\mathrm{op}}^2 = \|A^*A\|_{\mathrm{op}}$ and $(AB)^* = B^*A^*$ and the self-adjointness of $W(x)^{1/2}$ and $\langle W\rangle_S^{-1/2}$. By the previous exercises, we get

$$\begin{aligned}
\fint_S \|W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}^2\, \mathrm{d}x &= \fint_S \|(W(x)^{1/2}\langle W\rangle_S^{-1/2})^* W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}\, \mathrm{d}x \\
&= \fint_S \|((\langle W\rangle_S^{-1/2})^*(W(x)^{1/2})^* W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}\, \mathrm{d}x \\
&= \fint_S \|\langle W\rangle_S^{-1/2} W(x)\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}\, \mathrm{d}x \\
&\le \fint_S n\cdot \mathrm{tr}\left(\langle W\rangle_S^{-1/2} W(x)\langle W\rangle_S^{-1/2}\right)\mathrm{d}x \\
&= n\cdot \mathrm{tr}\fint \langle W\rangle_S^{-1/2} W(x)\langle W\rangle_S^{-1/2}\,\mathrm{d}x \\
&= n\cdot \mathrm{tr}\left(\langle W\rangle_S^{-1/2}\fint W(x)\,\mathrm{d}x\,\langle W\rangle_S^{-1/2}\right) \\
&= n\cdot \mathrm{tr}\,(I) \\
&= n^2.
\end{aligned}$$

□

Exercise 2.5.10. Suppose that $\mathscr{S}$ is *disjoint* and W is a general matrix weight (not necessarily in A_2). Show that in this case Lemma 2.5.4 holds with just a dimensional constant in place of $c_{d,n}[W]_{A_2}^{1/2}$.

Solution. We need to show that with these assumptions we have

$$\left(\sum_{S\in\mathscr{S}} \frac{1}{|S|}\left[\int_S \|W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}\,\psi(x)\,\mathrm{d}x\right]^2\right)^{1/2} \leq c_n\|\psi\|_{L^2(\mathbb{R}^d)}.$$

This is very simple with the help of Hölder's inequality (H) and the previous exercise:

$$\begin{aligned}\left(\sum_{S\in\mathscr{S}} \frac{1}{|S|}\left[\int_S \|W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}\,\psi(x)\,\mathrm{d}x\right]^2\right)^{1/2} &\overset{\text{(H)}}{\leq} \left(\sum_{S\in\mathscr{S}} \frac{1}{|S|}\int_S \|W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}^2\,\mathrm{d}x \int_S \psi(x)^2\,\mathrm{d}x\right)^{1/2}\\ &= \left(\sum_{S\in\mathscr{S}} \fint_S \|W(x)^{1/2}\langle W\rangle_S^{-1/2}\|_{\mathrm{op}}^2\,\mathrm{d}x \int_S \psi(x)^2\,\mathrm{d}x\right)^{1/2}\\ &\overset{\text{Ex. 2.5.9}}{\leq} c_n\left(\sum_{S\in\mathscr{S}}\int_S \psi(x)^2\,\mathrm{d}x\right)^{1/2}\\ &\overset{\mathscr{S}\text{ disjoint}}{\leq} c_n\|\psi\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

□

Exercise 2.5.11. Let $W \in A_2$ be a $\mathscr{L}(\mathbb{R}^n)$-valued matrix weight, and $x \in \mathbb{R}^n$ a non-zero vector. Prove that $(Wx \,|\, x) \in A_2$ is a scalar-valued A_2 weight and

$$[(Wx \,|\, x)]_{A_2} \;\leq\; [W]_{A_2}. \tag{1}$$

***Solution*.** Since $W \in A_2$ and $t \mapsto (W(t)y \,|\, y)$ is a scalar-valued function for every fixed $y \in \mathbb{R}^n$, we only need to show the estimate (1). First, let us fix $x \in \mathbb{R}^n \setminus \{0\}$ and set

$$f_Q(t) = 1_Q(t)\frac{(W(t)x \,|\, x)^{-1/2}}{|Q|^{1/2}}x$$

for every cube Q. These functions satisfy

$$\|f_Q\|^2_{L^2(W)} \;=\; \int (Wf_Q \,|\, f_Q) \;=\; ⨍_Q (Wx \,|\, x)^{-1/2}(Wx \,|\, x)^{-1/2}(Wx \,|\, x) \;=\; 1.$$

Let us also denote

$$T_Q g = 1_Q \langle g \rangle_Q.$$

By Proposition 2.2.1 (P) and the self-adjointness of W (S), we get

$$\begin{aligned}
[W]_{A_2} \;&=\; \sup_Q \|\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}\|^2_{\mathrm{op}} \\
&\overset{(\mathrm{P})}{=}\; \sup_Q \|T_Q\|^2_{\mathrm{op}} \\
&=\; \sup_Q \sup_{\|g\|_{L^2(W)}\leq 1} \|T_Q g\|^2_{L^2(W)} \\
&\geq\; \sup_Q \|T_Q f_Q\|^2_{L^2(W)} \\
&=\; \sup_Q \int_Q (W\langle f\rangle_Q \,|\, \langle f\rangle_Q) \\
&=\; \sup_Q \langle (Wx \,|\, x)^{-1/2}\rangle_Q^2 ⨍_Q (Wx \,|\, x) \\
&\overset{(\mathrm{S})}{=}\; \sup_Q ⨍_Q (Wx \,|\, x)^{-1} ⨍_Q (Wx \,|\, x) \\
&=\; [(Wx \,|\, x)]_{A_2}.
\end{aligned}$$

□

Exercise 2.5.12. Let $w \in A_2$ and $\sigma = w^{-1}$ be scalar-valued weights. Prove the linear bound

$$\|T_{\mathscr{S}}^{w,\sigma}\|_{L^2 \to L^2} \leq c_n [w]_{A_2}$$

in this case. Why does your argument not work for matrix weights? (Or, if it does, you have proved the matrix A_2 conjecture!)

Solution. Since w and σ are scalar-valued weights, we have

$$\begin{aligned} T_{\mathscr{S}}^{w,\sigma}\phi(x) &= \sum_{S \in \mathscr{S}} 1_S(x) \fint_S \|w(x)^{1/2}\sigma(y)^{1/2}\|_{\mathrm{op}}\, \phi(y)\, \mathrm{d}y \\ &= \sum_{S \in \mathscr{S}} 1_S(x) \fint_S w(x)^{1/2}\sigma(y)^{1/2}\, \phi(y)\, \mathrm{d}y \\ &= \sum_{S \in \mathscr{S}} 1_S(x)\, w(x)^{1/2} \langle \phi\, \sigma^{1/2} \rangle_S. \end{aligned}$$

Thus, by Proposition 1.2.2 (P), we get

$$\begin{aligned} \|T_{\mathscr{S}}^{w,\sigma}\phi\|_{L^2(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left(\sum_{S \in \mathscr{S}} 1_S(x)\, w(x)^{1/2}\, \langle \phi\, \sigma^{1/2} \rangle_S \right)^2 \mathrm{d}x \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d} \left(\sum_{S \in \mathscr{S}} 1_S(x)\, \langle \phi\, \sigma^{1/2} \rangle_S \right)^2 w(x)\, \mathrm{d}x \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d} T_{\mathscr{S}}(\phi\, \sigma^{1/2})(x)^2\, w(x)\, \mathrm{d}x \right)^{1/2} \\ &= \|T_{\mathscr{S}}(\phi\, \sigma^{1/2})\|_{L^2(w)} \\ &\overset{\text{(P)}}{\leq} 4\gamma^{-1} [w]_{A_2^{\mathscr{D}}} \|\phi\, \sigma^{1/2}\|_{L^2(w)} \\ &= 4\gamma^{-1} [w]_{A_2}^{\mathscr{D}} \left(\int_{\mathbb{R}^d} \phi^2(x)\, \sigma(x)\, w(x)\, \mathrm{d}x \right)^{1/2} \\ &= 4\gamma^{-1} [w]_{A_2}^{\mathscr{D}} \left(\int_{\mathbb{R}^d} \phi^2(x)\, \mathrm{d}x \right)^{1/2} \\ &= 4\gamma^{-1} [w]_{A_2}^{\mathscr{D}} \|\phi\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where γ is the sparseness parameter of the collection $\mathscr{S}$ and $[w]_{A_2}^{\mathscr{D}} \leq [w]_{A_2}$.

The argument does not work for matrix weights for several reasons, the most obvious being that breaking $\|W(x)^{1/2}\Sigma(y)^{1/2}\|_{\mathrm{op}} \leq \|W(x)^{1/2}\|_{\mathrm{op}} \|\Sigma(y)^{1/2}\|_{\mathrm{op}}$ cannot be reversed later[1]. □

[1] We notice that $1 = \|I\|_{\mathrm{op}} = \|AA^{-1}\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}} \|A^{-1}\|_{\mathrm{op}}$ for any invertible matrix. The right-hand side can be arbitrarily large which can be seen by choosing

$$A = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1/M & 0 \\ 0 & 1 \end{bmatrix},$$

and noticing that $\|A\|_{\mathrm{op}} = M$ and $\|A^{-1}\|_{\mathrm{op}} = 1$ for any $M \geq 1$ by Exercise 2.5.8.