Dyadic analysis and weights, Spring 2017
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Solutions to the exercise set 4 (6 pages)

Exercise 1.9.8. Show the following converse of Theorem 1.9.4: If a weight w satisfies the reverse Hölder inequality

$$\left(\fint_Q w^{1+\varepsilon}\right)^{1/(1+\varepsilon)} \leq K \fint_Q w$$

for all $Q \in \mathscr{D}$, then $w \in A_\infty^{\mathscr{D}}$. Estimate $[w]_{A_\infty}^{\mathscr{D}}$ in terms of K and ε.

Solution. Suppose that $Q \in \mathscr{D}$. By Hölder's inequality and the L^p-boundedness of the dyadic Hardy-Littlewood maximal operator (Corollary 1.1.2), we have

$$\begin{aligned}
\int_Q M_Q w &\leq \left(\int_Q (M_Q w)^{1+\varepsilon}\right)^{1/(1+\varepsilon)} \left(\int_Q 1_Q\right)^{1/(1+\varepsilon)'} \\
&\leq (1+\varepsilon)' \left(\int_Q w^{1+\varepsilon}\right)^{1/(1+\varepsilon)} |Q|^{1/(1+\varepsilon)'} \\
&= (1+\varepsilon)' \left(\fint_Q w^{1+\varepsilon}\right)^{1/(1+\varepsilon)} |Q|^{1/(1+\varepsilon)} |Q|^{1/(1+\varepsilon)'} \\
&= (1+\varepsilon)' \left(\fint_Q w^{1+\varepsilon}\right)^{1/(1+\varepsilon)} |Q| \\
&\leq (1+\varepsilon)' K \left(\fint_Q w\right) |Q| \\
&= (1+\varepsilon)' K \int_Q w.
\end{aligned}$$

Thus, we have $w \in A_\infty^{\mathscr{D}}$ and $[w]_{A_\infty}^{\mathscr{D}} \leq (1+\varepsilon)' K = \frac{1+\varepsilon}{\varepsilon} K$. □

Exercise 1.9.9. Consider the following truncated version of M_Q:

$$M_Q^N f(x) := \sup_{\substack{Q' \in \mathscr{D}, Q' \subseteq Q \\ \ell(Q') \geq 2^{-N}\ell(Q)}} 1_{Q'}(x) \langle |f| \rangle_{Q'},$$

and define the truncated A_∞ constant as the smallest constant in the following inequality:

$$\int_Q M_Q^N w \;\leq\; [w]_{A_\infty}^{\mathscr{D},N} \int_Q w \quad \forall Q \in \mathscr{D}.$$

Show that $[w]_{A_\infty}^{\mathscr{D},N} < \infty$ for any weight w, and that $[w]_{A_\infty}^{\mathscr{D},N} \to [w]_{A_\infty}^{\mathscr{D}}$ as $N \to \infty$.

Solution. For every $Q \in \mathscr{D}$ we have

$$\begin{aligned}
\int_Q M_Q^N w \;=\; \int_Q \sup_{\substack{Q' \in \mathscr{D}, Q' \subseteq Q \\ \ell(Q') \geq 2^{-N}\ell(Q)}} 1_{Q'}(x) \langle w \rangle_{Q'} \;&=\; \int_Q \sup_{\substack{Q' \in \mathscr{D}, Q' \subseteq Q \\ \ell(Q') \geq 2^{-N}\ell(Q)}} \frac{1_{Q'}(x)}{|Q'|} w(Q') \\
&\leq\; \frac{w(Q)}{|2^{-N}Q|} \int_Q \sup_{\substack{Q' \in \mathscr{D}, Q' \subseteq Q \\ \ell(Q') \geq 2^{-N}\ell(Q)}} 1_{Q'}(x) \\
&=\; \frac{w(Q)}{2^{-Nd}|Q|} |Q| \\
&=\; 2^{Nd} \int_Q w.
\end{aligned}$$

Thus, $[w]_{A_\infty}^{\mathscr{D},N} \leq 2^{Nd} < \infty$ for every weight w.

Let us then show that $[w]_{A_\infty}^{\mathscr{D},N} \to [w]_{A_\infty}^{\mathscr{D}}$. We first notice that since the values of $M_Q^N w$ increase pointwise as N increases, we can write $M_Q w(x) = \lim_{N\to\infty} M_Q^N w(x)$ for every x. In particular, the sequence $([w]_{A_\infty}^{\mathscr{D},N})_{N=0}^{\infty}$ is increasing and thus, the limit $\lim_{N\to\infty} [w]_{A_\infty}^{\mathscr{D},N}$ exists (it may be ∞). By definition, we have $[w]_{A_\infty}^{\mathscr{D},N} \leq [w]_{A_\infty}^{\mathscr{D}}$ for every N and hence, $\lim_{N\to\infty} [w]_{A_\infty}^{\mathscr{D},N} \leq [w]_{A_\infty}^{\mathscr{D}}$. Since we also have

$$\int_Q M_Q w \;=\; \int_Q \lim_{N\to\infty} M_Q^N w \;=\; \lim_{N\to\infty} \int_Q M_Q^N w \;\leq\; \lim_{N\to\infty} [w]_{A_\infty}^{\mathscr{D},N} \int_Q w$$

by the monotone convergence theorem, we have $\lim_{N\to\infty} [w]_{A_\infty}^{\mathscr{D},N} \geq [w]_{A_\infty}^{\mathscr{D}}$ as $[w]_{A_\infty}^{\mathscr{D}}$ is the smallest constant C in the inequality $\int_Q M_Q w \leq C \int_Q w$. Hence, $[w]_{A_\infty}^{\mathscr{D},N} \to [w]_{A_\infty}^{\mathscr{D}}$ as $N \to \infty$. □

Exercise 1.9.10. The following condition is often used as the definition of the (dyadic) A_∞: There are constant $\delta, \eta \in (0,1)$ such that for all (dyadic) cubes Q and all measurable subsets $E \subset Q$, if $|E| \le \delta|Q|$, then $w(E) \le \eta w(Q)$. Prove that this condition implies the dyadic A_∞ condition as we have defined it.

Solution. For every $\lambda \ge 0$, we denote $E := E_\lambda := \{x \in Q_0 \colon M_Q^N w(x) > \lambda\}$. By the same considerations as in the proof of Theorem 1.1.1, we have $E = \bigcup_{Q \in \mathcal{F}^*} Q$ for a collection $\mathcal{F}^* := \mathcal{F}^*_\lambda$ of maximal disjoint cubes $Q \subseteq Q_0$ such that $\ell(Q) \ge 2^{-N}\ell(Q_0)$. Let us start by making two observations.

i) For any cube $Q \in \mathcal{F}^*_\lambda$, we have

$$\langle w\rangle_Q > \lambda \ge \sup_{\substack{Q' \supsetneq Q,\, Q' \subseteq Q_0, \\ \ell(Q') \ge 2^{-N}\ell(Q_0)}} \langle w\rangle_{Q'}$$

and thus, for any point $x \in Q$ we get

$$M_{Q_0}^N w(x) = \sup_{\substack{Q' \ni x,\, Q' \subseteq Q_0, \\ \ell(Q') \ge 2^{-N}\ell(Q_0)}} \langle w\rangle_{Q'} = \sup_{\substack{Q' \ni x,\, Q' \subseteq Q, \\ \ell(Q') \ge 2^{-N}\ell(Q_0)}} \langle w\rangle_{Q'} \le \sup_{\substack{Q' \ni x,\, Q' \subseteq Q, \\ \ell(Q') \ge 2^{-N}\ell(Q)}} \langle w\rangle_{Q'} = M_Q^N w(x). \tag{1}$$

ii) Recall that we have $M_{\mathscr{D}} \colon L^1 \to L^{1,\infty}$ by Theorem 1.1.1. Since $M_{Q_0}^N w(x) \le M_{\mathscr{D}}(1_{Q_0} w)(x)$ for every x, we have

$$|E| \le \left|\{x \in \mathbb{R}^d \colon M_{\mathscr{D}}(1_{Q_0} w)(x) > \lambda\}\right| \le \frac{\|1_{Q_0} w\|_{L^1}}{\lambda} = \frac{w(Q_0)}{\lambda} = \frac{\langle w\rangle_{Q_0}|Q_0|}{\lambda}. \tag{2}$$

Let us choose $\lambda = \frac{\langle w\rangle_{Q_0}}{\delta}$. Then, by (2), we have $|E| \le \delta|Q_0|$ and thus, by assumption, we have $w(E) \le \eta w(Q_0)$. Hence, we get

$$\begin{aligned}
\int_E M_{Q_0}^N w &= \sum_{Q \in \mathcal{F}^*} \int_Q M_{Q_0}^N w \overset{(1)}{\le} \sum_{Q \in \mathcal{F}^*} \int_Q M_Q^N w \\
&\le [w]_{A_\infty}^{\mathscr{D},N} \sum_{Q \in \mathcal{F}^*} \int_Q w \\
&= [w]_{A_\infty}^{\mathscr{D},N} w(E) \\
&\le \eta [w]_{A_\infty}^{\mathscr{D},N} w(Q_0)
\end{aligned}$$

and

$$\int_{Q_0 \setminus E} M_{Q_0}^N w \le \int_{Q_0 \setminus E} \frac{\langle w\rangle_{Q_0}}{\delta} \le \int_{Q_0} \frac{\langle w\rangle_{Q_0}}{\delta} = \frac{1}{\delta} w(Q_0).$$

Combining these calculations gives us $\int_{Q_0} M_{Q_0}^N w \le (\eta[w]_{A_\infty}^{\mathscr{D},N} + \frac{1}{\delta}) w(Q_0)$. In particular, by the definition of the constant $[w]_{A_\infty}^{\mathscr{D},N}$, we have $[w]_{A_\infty}^{\mathscr{D},N} \le \eta[w]_{A_\infty}^{\mathscr{D},N} + \frac{1}{\delta}$. Now we can use Exercise 1.9.9:

- Since $[w]_{A_\infty}^{\mathscr{D},N} < \infty$ for every weight w, we get

$$[w]_{A_\infty}^{\mathscr{D},N} \le \eta[w]_{A_\infty}^{\mathscr{D},N} + \frac{1}{\delta} \quad \Longleftrightarrow \quad [w]_{A_\infty}^{\mathscr{D},N} \le \frac{1}{\delta(1-\eta)}$$

  for every $N \in \mathbb{N}$.

- Since $[w]_{A_\infty}^{\mathscr{D},N} \to [w]_{A_\infty}^{\mathscr{D}}$ as $N \to \infty$, we have

$$[w]_{A_\infty}^{\mathscr{D}} = \lim_{N\to\infty} [w]_{A_\infty}^{\mathscr{D},N} \le \lim_{N\to\infty} \frac{1}{\delta(1-\eta)} = \frac{1}{\delta(1-\eta)} < \infty.$$

Thus, since $[w]_{A_\infty}^{\mathscr{D}} < \infty$, we have $w \in A_\infty^{\mathscr{D}}$. □

Exercise 2.2.2. For self-adjoint matrices A, B, we introduce the partial order $\leq$ as follows:

$$A \leq B \quad \overset{\text{def}}{\Longleftrightarrow} \quad (Ax|x) \leq (Bx|x) \ \ \forall x \in \mathbb{C}^n.$$

For all positive matrices A, B, show that

$$A \leq B \quad \Longleftrightarrow \quad \|A^{1/2}B^{-1/2}\|_{\text{op}} \leq 1 \quad \Longleftrightarrow \quad \|B^{-1/2}A^{1/2}\|_{\text{op}} \leq 1 \quad \Longleftrightarrow \quad B^{-1} \leq A^{-1},$$

i.e., all four listed conditions are equivalent.

Solution. For simplicity, we denote

(A) $A \leq B$,

(B) $\|A^{1/2}B^{-1/2}\|_{\text{op}} \leq 1$,

(C) $\|B^{-1/2}A^{1/2}\|_{\text{op}} \leq 1$,

(D) $B^{-1} \leq A^{-1}$.

Since the proofs of (A) $\Leftrightarrow$ (B) and (C) $\Leftrightarrow$ (D) are virtually the same, we will just prove that (A) $\Leftrightarrow$ (B) and (B) $\Leftrightarrow$ (C).

(A) $\Rightarrow$ (B) Since $(Ay\,|\,y) \leq (By\,|\,y)$ for every y, for every x such that $\|x\| \leq 1$ we have

$$\begin{aligned} \|A^{1/2}B^{-1/2}x\|^2 &= \left(A^{1/2}B^{-1/2}x\,|\,A^{1/2}B^{-1/2}x\right) = \left(AB^{-1/2}x\,|\,B^{-1/2}x\right) \\ &\leq \left(BB^{-1/2}x\,|\,B^{-1/2}x\right) \\ &= (x\,|\,x) = \|x\| \leq 1. \end{aligned}$$

Thus, $\|A^{1/2}B^{-1/2}\|_{\text{op}} \leq 1$.

(B) $\Rightarrow$ (A) Since $\|A^{1/2}B^{-1/2}\|_{\text{op}} \leq 1$, for every x we get

$$\begin{aligned} (Ax\,|\,x) = \left(A^{1/2}x\,|\,A^{1/2}x\right) = \|A^{1/2}x\|^2 &= \|A^{1/2}B^{-1/2}B^{1/2}x\|^2 \\ &\leq \|A^{1/2}B^{-1/2}\|_{\text{op}}^2\|B^{1/2}x\|^2 \\ &\leq \|B^{1/2}x\|^2 = \left(B^{1/2}x\,|\,B^{1/2}x\right) = (Bx\,|\,x). \end{aligned}$$

Thus, $A \leq B$.

(B) $\Leftrightarrow$ (C) To show this equivalence, we only need to recall that $\|T\|_{\text{op}} = \|T^*\|_{\text{op}}$ and $(ST)^* = T^*S^*$ for any bounded linear operators on a complex Hilbert space. Indeed, since the matrices $A^{1/2}$ and $B^{1/2}$ are self-adjoint, we have

$$\|A^{1/2}B^{-1/2}\|_{\text{op}} = \|(A^{1/2}B^{-1/2})^*\|_{\text{op}} = \|(B^{-1/2})^*(A^{1/2})^*\|_{\text{op}} = \|B^{-1/2}A^{1/2}\|_{\text{op}}$$

and the equivalence of (B) and (C) follows immediately.

Hence, (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) $\Leftrightarrow$ (D). $\square$

Exercise 2.2.3. Show that $W \in A_2$ if and only if $\langle W\rangle_Q \le C\langle W^{-1}\rangle_Q^{-1}$, if and only if $W^{-1} \in A_2$, and the optimal constant satisfies $C = [W]_{A_2} = [W^{-1}]_{A_2}$.

Solution. Recall that A_2 was the set of matrix weights W such that $[W]_{A_2} := \sup_Q \|\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}\|_{\mathrm{op}}^2 < \infty$. For simplicity, let us denote

(A) $W \in A_2$,

(B) $\langle W\rangle_Q \le C\langle W^{-1}\rangle_Q^{-1}$ for all cubes Q,

(C) $W^{-1} \in A_2$.

We will prove the claim in three parts:

(A) $\Rightarrow$ (B) For every $x \in \mathbb{R}^d$ we have

$$\begin{aligned}(\langle W\rangle_Q x|x) \;=\; \|\langle W\rangle_Q^{1/2}x\|^2 &= \|\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{-1/2}x\| \\ &\le \|\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}\|_{\mathrm{op}}^2\|\langle W^{-1}\rangle_Q^{-1/2}x\| \\ &\overset{(A)}{\le} [W]_{A_2}\|\langle W^{-1}\rangle_Q^{-1/2}x\| \\ &= [W]_{A_2}(\langle W^{-1}\rangle_Q x|x).\end{aligned}$$

Thus, (B) holds for $C = [W]_{A_2}$.

(B) $\Rightarrow$ (A) Suppose that Q is a cube and let $x \in \mathbb{R}^d$, $\|x\| \le 1$. We get

$$\begin{aligned}\|\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}x\|^2 &= (\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}x|\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}x) \\ &= (\langle W\rangle_Q\langle W^{-1}\rangle_Q^{1/2}x|\langle W^{-1}\rangle_Q^{1/2}x) \\ &\overset{(B)}{\le} (C\langle W^{-1}\rangle_Q^{-1}\langle W^{-1}\rangle_Q^{1/2}x|\langle W^{-1}\rangle_Q^{1/2}x) \\ &= C(x|x) \;\le\; C.\end{aligned}$$

Thus, $[W]_{A_2} \le C < \infty$ and hence, $W \in A_2$.

(C) $\Leftrightarrow$ (B) Using the previous part of the proof and the previous exercise, we get

$$\begin{aligned}\text{(C)} &\overset{(A)\Leftrightarrow(B)}{\iff} \langle W^{-1}\rangle_Q \le C\langle (W^{-1})^{-1}\rangle_Q^{-1} = C\langle W\rangle_Q^{-1} \quad \text{for all cubes } Q \\ &\overset{\text{Ex. 2.2.2.}}{\iff} (C\langle W\rangle_Q^{-1})^{-1} \le \langle W^{-1}\rangle_Q^{-1} \quad \text{for all cubes } Q \\ &\iff \langle W\rangle_Q^{-1} \le C\langle W^{-1}\rangle_Q^{-1} \quad \text{for all cubes } Q \\ &\iff \text{(B)},\end{aligned}$$

where $C = [W^{-1}]_{A_2}$.

Hence, (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C). □

Exercise 2.2.4. Show that any matrix weight W satisfies estimate

$$\langle W^{-1}\rangle_Q^{-1} \leq \langle W\rangle_Q.$$

Solution. Suppose that $x \in \mathbb{R}^d$. First, we notice that by the general Cauchy-Schwarz (C1) and the L^2-Cauchy-Schwarz (C2) we get

$$\begin{aligned}
\Big(\langle W^{-1}\rangle_Q^{-1}x \,|\, x\Big) &= \Big\langle \Big(W^{-1/2}\langle W^{-1}\rangle_Q^{-1}x \,|\, W^{1/2}x\Big)\Big\rangle_Q \\
&\overset{(C1)}{\leq} \Big\langle \|W^{-1/2}\langle W^{-1}\rangle_Q^{-1}x\| \, \|W^{1/2}x\|\Big\rangle_Q \\
&\overset{(C2)}{\leq} \Big\langle \|W^{-1/2}\langle W^{-1}\rangle_Q^{-1}x\|^2\Big\rangle_Q^{1/2} \Big\langle \|W^{1/2}x\|\Big\rangle_Q^{1/2}.
\end{aligned}$$

Since we have

$$\begin{aligned}
\Big\langle \|W^{-1/2}\langle W^{-1}\rangle_Q^{-1}x\|^2\Big\rangle_Q^{1/2} &= \Big\langle \Big(W^{-1/2}\langle W^{-1}\rangle_Q^{-1}x \,|\, W^{-1/2}\langle W^{-1}\rangle_Q^{-1}x\Big)\Big\rangle_Q^{1/2} \\
&= \Big\langle \Big(W^{-1}\langle W^{-1}\rangle_Q^{-1}x \,|\, \langle W^{-1}\rangle_Q^{-1}x\Big)\Big\rangle_Q^{1/2} \\
&= \Big(\langle W^{-1}\rangle_Q\langle W^{-1}\rangle_Q^{-1}x \,|\, \langle W^{-1}\rangle_Q^{-1}x\Big)^{1/2} \\
&= \big(\langle W^{-1}\rangle_Q x \,|\, x\big)^{1/2}.
\end{aligned}$$

and

$$\Big\langle \|W^{1/2}x\|\Big\rangle_Q^{1/2} = \Big\langle \Big(W^{1/2}x \,|\, W^{1/2}x\Big)\Big\rangle_Q^{1/2} = \langle (Wx \,|\, x)\rangle_Q^{1/2} = \Big(\langle W\rangle_Q \, x \,|\, x\Big)^{1/2},$$

we have proven

$$\Big(\langle W^{-1}\rangle_Q^{-1}x \,|\, x\Big) \leq \big(\langle W^{-1}\rangle_Q x \,|\, x\big)^{1/2} \Big(\langle W\rangle_Q \, x \,|\, x\Big)^{1/2} \quad\iff\quad \Big(\langle W^{-1}\rangle_Q^{-1}x \,|\, x\Big) \leq \Big(\langle W\rangle_Q \, x \,|\, x\Big).$$

Hence, $\langle W^{-1}\rangle_Q^{-1} \leq \langle W\rangle_Q$. $\square$