Dyadic analysis and weights, Spring 2017
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Solutions to the exercise set 3 (8 pages)

Exercise 1.8.13. Suppose that $V^r Sf(x) < \infty$ at some point x. Show that $\lim_{\varepsilon \to 0} S_\varepsilon f(x)$ exists at this point.

Solution. Recall that given a family of linear operators $(S_\varepsilon)_{\varepsilon \in (0,\infty)}$, we define

$$\begin{aligned} V_\varepsilon^r Sf(x) &:= \sup \left(\sum_{j=1}^N \left| S_{\varepsilon_{j-1}} f(x) - S_{\varepsilon_j} f(x) \right|^r \right)^{1/r}, \\ V^r Sf(x) &:= V_0^r Sf(x), \end{aligned}$$

where the supremum is taken over all increasing sequences $\varepsilon \le \varepsilon_0 \le \ldots \le \varepsilon_N$ (with the additional requirement that $0 < \varepsilon_0$ if $\varepsilon = 0$), where N is finite but arbitary.

Since $\mathbb{R}$ is complete, it suffices to fix a sequence $(y_n)_{n=1}^\infty$ such that $y_n \searrow 0$ and show that the corresponding sequence $(z_n)_{n=1}^\infty$, $z_n = S_{y_n} f(x)$, is Cauchy. Suppose that $\varepsilon > 0$. Let us set $N_\varepsilon := \lceil (V^r Sf(x)/\varepsilon)^r \rceil$. Since $V^r Sf(x) < \infty$, we have $N_\varepsilon \in \mathbb{N}$. We now claim that there exist at most N_ε disjoint intervals $[a_i, b_i) \subset (0,\infty)$ such that $\left| S_{a_j} f(x) - S_{b_j} f(x) \right| \ge \varepsilon$:

> If no such intervals exist, we are done. Otherwise, choose any such interval $I_1 = [a_1, b_1)$, and consider the set $(0,\infty) \setminus I_1$. If possible, choose another such interval $I_2 = [a_2, b_2) \subset (0,\infty) \setminus I_1$, and continue the process for the set $(0,\infty) \setminus (I_1 \cup I_2)$. For contradiction, suppose that we can choose $N_\varepsilon + 1$ intervals this way. Then we have
>
> $$\begin{aligned} V^r Sf(x) &= \sup_{\substack{N \in \mathbb{N}, \\ 0 < \varepsilon_j \le \varepsilon_{j+1}}} \left(\sum_{j=1}^N \left| S_{\varepsilon_{j-1}} f(x) - S_{\varepsilon_j} f(x) \right|^r \right)^{1/r} \\ &\ge \left(\sum_{j=1}^{N_\varepsilon + 1} \left| S_{b_j} f(x) - S_{a_j} f(x) \right|^r \right)^{1/r} \\ &\ge \left(\sum_{j=1}^{N_\varepsilon + 1} \varepsilon^r \right)^{1/r} \\ &= (N_\varepsilon + 1)^{1/r} \varepsilon \;>\; N_\varepsilon^{1/r} \varepsilon \;\ge\; \left(\frac{V^r Sf(x)^r}{\varepsilon^r} \right)^{1/r} \varepsilon \;=\; V^r Sf(x), \end{aligned}$$
>
> which is a contradiction.

Since the number of these intervals $[a_j, b_j)$ is finite, we may choose $n_\varepsilon \in \mathbb{N}$ to be so large that $0 < y_n < \min_j a_j$ for every $n \ge n_\varepsilon$. Thus, for any $n, m \ge n_\varepsilon$ we have $0 < y_n, y_m < \min_j a_j$ and hence, $|S_{y_n} f(x) - S_{y_m} f(x)| < \varepsilon$. In particular, the sequence (z_n) is Cauchy. □

Exercise 1.8.14. Check that if $f \in \bigcup_{p\in[1,\infty)} L^p(\mathbb{R}^d)$, then both $A_\varepsilon f(x)$ and $T_\varepsilon f(x)$ tend to zero as $\varepsilon \to \infty$.

Solution. Recall that

$$A_\varepsilon f(x) \;:=\; \fint_{B(x,\varepsilon)} f(y)\,\mathrm{d}y, \qquad\qquad T_\varepsilon f(x) \;:=\; \int_{|x-y|>\varepsilon} K(x,y)f(y)\mathrm{d}y.$$

Let us fix $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^d)$. If $p = 1$, we have $p' = \infty$ and interpret $1/p' = 0$. By Hölder's inequality (H), we get

$$|A_\varepsilon f(x)| \leq \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} |f(y)|\,\mathrm{d}y \overset{(\mathrm{H})}{\leq} \frac{1}{|B(x,\varepsilon)|}\|f\|_{L^p}|B(x,\varepsilon)|^{1/p'} = \frac{\|f\|_{L^p}}{|B(x,\varepsilon)|^{1/p}} = \frac{\|f\|_{L^p}}{c_d^{1/p}\varepsilon^{d/p}} \xrightarrow{\varepsilon\to\infty} 0.$$

For T_ε, let consider the case $p = 1$ separately.

<u>$p = 1$</u>: In this case, we can simply use the size property of the Calderón-Zygmund kernel K (CZ):

$$|T_\varepsilon f(x)| \;\leq\; \int_{|x-y|>\varepsilon} |K(x,y)||f(y)|\,\mathrm{d}y \;\overset{(\mathrm{CZ})}{\leq}\; \int_{|x-y|>\varepsilon} \frac{|f(y)|}{|x-y|^d}\,\mathrm{d}y \;\leq\; \frac{1}{\varepsilon^d}\|f\|_{L^1} \xrightarrow{\varepsilon\to\infty} 0$$

<u>$p > 1$</u>: In this case, we need to be a little more careful. We recall from real analysis that for $a > 0$ the function $x \mapsto 1/|x|^a$ is integrable over $\mathbb{R}^d \setminus B(x,\varepsilon)$ if and only if $a > d$. In particular, the function $x \mapsto 1/|x|^{dp'}$ is integrable over $\mathbb{R}^d \setminus B(x,\varepsilon)$. Thus, since $1_{|x-\cdot|>\varepsilon}(y) \searrow 0$ for all $y \in \mathbb{R}^d$ as $\varepsilon \searrow 0$, the size property of the Calderón-Zygmund kernel K (CZ), Hölder's inequality (H) and the dominated convergence theorem (DCT) give us

$$\begin{aligned} |T_\varepsilon f(x)| \;\leq\; \int_{|x-y|>\varepsilon} |K(x,y)||f(y)|\,\mathrm{d}y \;&\overset{(\mathrm{CZ})}{\leq}\; \int_{|x-y|>\varepsilon} \frac{|f(y)|}{|x-y|^d}\,\mathrm{d}y \\ &\overset{(\mathrm{H})}{\leq} \left(\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^{dp'}}\,\mathrm{d}y\right)^{1/p'} \|f\|_{L^p} \\ &\underset{(\mathrm{DCT})}{\xrightarrow{\varepsilon\to\infty}} 0. \end{aligned}$$

□

Exercise 1.8.15. For $0 < a < b$, prove that

$$V_a^r Tf(x) \ \le \ V_b^r Tf(x) + c_d c_K (1 + \log(b/a)) Mf(x).$$

Solution. First, we notice that

$$\begin{aligned}
V_a^r Tf(x) \ &= \ \sup \left(\sum_{i=1}^{N} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right|^r \right)^{1/r} \\
&= \ \sup \left(\sum_{i:\, a \le \varepsilon_{i-1} \le \varepsilon_i \le b} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right|^r + \sum_{i:\, \varepsilon_{i-1} > b} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right|^r \right)^{1/r} \\
&\le \ \sup \left(\sum_{i:\, a \le \varepsilon_{i-1} \le \varepsilon_i \le b} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right|^r \right)^{1/r} + \sup \left(\sum_{i:\, \varepsilon_{i-1} > b} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right|^r \right)^{1/r} \\
&= \ \sup \left(\sum_{i:\, a \le \varepsilon_{i-1} \le \varepsilon_i \le b} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right|^r \right)^{1/r} + V_b^r f(x) \ =: \ I + V_b^r f(x),
\end{aligned}$$

so we only need to show that $I \le c_d c_K (1 + \log(b/a)) Mf(x)$. For this, we use the size property of the Calderón-Zygmund kernels:

$$\begin{aligned}
I \ &\le \ \sup \left(\sum_{i:\, a \le \varepsilon_{i-1} \le \varepsilon_i \le b} \left| T_{\varepsilon_{i-1}} f(x) - T_{\varepsilon_i} f(x) \right| \right) \\
&= \ \sup \sum_{i:\, a \le \varepsilon_{i-1} \le \varepsilon_i \le b} \int_{\varepsilon_{i-1} < |x-y| < \varepsilon_i} |K(x,y)||f(y)| \, \mathrm{d}y \\
&\le \ \int_{a < |x-y| < b} |K(x,y)||f(y)| \, \mathrm{d}y \\
&\le \ c_K \int_{a < |x-y| < b} \frac{|f(y)|}{|x-y|^d} \, \mathrm{d}y \\
&= \ c_K \sum_{k \ge 0:\, 2^k a < b} \int_{2^k a < |x-y| < 2^{k+1} a} \frac{|f(y)|}{|x-y|^d} \, \mathrm{d}y \\
&\le \ c_K \sum_{k \ge 0:\, 2^k a < b} \frac{1}{(2^k a)^d} \int_{B(x, 2^{k+1} a)} |f(y)| \, \mathrm{d}y \\
&\le \ c_K c_d \sum_{k \ge 0:\, 2^k a < b} \fint_{B(x, 2^{k+1} a)} |f(y)| \, \mathrm{d}y \ \le \ \sum_{k \ge 0:\, 2^k a < b} c_K c_d Mf(x). \qquad (1)
\end{aligned}$$

Since we have

$$2^k a \le b \quad \Longrightarrow \quad k \le \log_2(b/a) \le 2 \log(b/a),$$

we get

$$\sum_{k \ge 0:\, 2^k a < b} 1 \ \le \ 1 + \sum_{k=1}^{\lceil 2 \log(b/a) \rceil} 1 \ \le \ 2 \left(1 + \log(b/a) \right).$$

Hence, we have proven the claim. $\square$

Remark 2. We note that the same proof gives us a slightly stronger result: we have

$$V_a^r Tf(x) \ \le \ V_b^r Tf(x) + c_d c_K (1 + \log(b/a)) Mf(x')$$

for every x' such that $|x - x'| < 2a$; we only need to replace $Mf(x)$ by $Mf(x')$ on the line (1).

Exercise 1.8.16. Define $\widetilde{V}^r T$ in a way analogous to $\widetilde{V}^r A$. Prove a pointwise bound for $\widetilde{V}^r T f$, which allows to conclude that $\widetilde{V}^r T \colon L^1 \to L^{1,\infty}$.

Solution. We define $\widetilde{V}^r T$ by setting

$$\widetilde{V}^r T f(x) \coloneqq \sup_{z \in \mathbb{R}^d} V^r_{|z-x|} T f(z),$$

and we claim that

$$\widetilde{V}^r T f(x) \leq c_d(\|\omega\|_{\mathrm{Dini}} + c_K) M f(x) + V^r T f(x) + c_d c_K \widetilde{V}^r A|f|(x). \tag{3}$$

This bound is straightforward to prove with the help of Remark 2 and Lemma 1.8.6. First, we apply Remark 2 with the choices $a = |z-x|$ and $b = 2|z-x|$:

$$V^r_{|z-x|} T f(x) \ \leq \ V^r_{2|z-x|} T f(z) + c_d c_K M f(x).$$

Then, we notice that

$$\begin{aligned} V^r_{2|z-x|} T f(z) \ &\leq \ \left| V^r_{2|z-x|} T f(z) - V^r_{2|z-x|} T f(x) \right| + V^r_{2|z-x|} T f(x) \\ &\leq \ \left| V^r_{2|z-x|} T f(z) - V^r_{2|z-x|} T f(x) \right| + V^r T f(x), \end{aligned}$$

and since $|z-x| \leq 2|z-x|/2$, we can apply Lemma 1.8.6 for the first term:

$$\left| V^r_{2|z-x|} T f(z) - V^r_{2|z-x|} T f(x) \right| \ \leq \ c_d \left(\|\omega\|_{\mathrm{Dini}} + c_K \right) M f(x) + c_d c_K \widetilde{V} A|f|(x).$$

Combining the previous estimates gives us the bound (3). By Theorem 1.8.3 and Theorem 1.8.4, this bound is enough to conclude that $\widetilde{V}^r T \colon L^1 \to L^{1,\infty}$. □

Exercise 1.8.17. Prove a pointwise bound for $M_{V^rA}f$, which allows to conclude that $M_{V^rA}\colon L^1 \to L^{1,\infty}$ (and hence to apply Lerner's theorem to V^rA).

Solution. Recall that the function $M_{V^rA}f$ is defined as

$$M_{V^rA}f(x) = \sup_{Q\ni x}\sup_{z\in Q} V^rA(1_{(3Q)^c}f)(z).$$

We claim that we have the pointwise bound

$$M_{V^rA}f(x) \;\le\; \widetilde{V}^rAf(x) + c_dMf(x). \tag{4}$$

We can prove this bound using the same techniques that we used in the proof of Lemma 1.8.9. We denote

$$\widetilde{f} := 1_{(3Q)^c}f, \qquad v_i := v_{\varepsilon_i,\varepsilon_{i+1}} := \left|A_{\varepsilon_i}\widetilde{f}(z) - A_{\varepsilon_{i+1}}\widetilde{f}(z)\right|$$

and with this notation we have

$$V^rA\widetilde{f}(z) \;=\; \sup\left(\sum_i v_i^r\right)^{1/r}.$$

Using the same considerations as in the proof of Lemma 1.8.9, we get

$$\sup\left(\sum_i v_i^r\right)^{1/r} \;\le\; \left(\sup_{\varepsilon_N\le \ell(Q)} + \sup_{\ell(Q)\le\varepsilon_0\le\varepsilon_N\le 2\sqrt{d}\ell(Q)} + \sup_{2\sqrt{d}\ell(Q)\le\varepsilon_0}\right)\left(\sum_i v_i^r\right)^{1/r} \;=:\; I + II + III.$$

Since $B(z,r)\subset 3Q$ for every $r\le \ell(Q)$, we have $I=0$. Also, we notice that since $|x-z|\le 2\sqrt{d}\ell(Q)$, we get

$$III \;\le\; \sup_{|x-z|\le\varepsilon_0}\left(\sum_i v_i^r\right)^{1/r} \;\le\; \sup_{z\in\mathbb{R}^d}\sup_{|x-z|\le\varepsilon_0}\left(\sum_i v_i^r\right)^{1/r} \;=\; \sup_{z\in\mathbb{R}^d} V^r_{|z-x|}Af(z) \;=\; \widetilde{V}^rAf(x).$$

Thus, we only need to find a suitable bound for II. Suppose that $\ell(Q)\le\varepsilon_i\le\varepsilon_{i+1}\le 2\sqrt{d}\ell(Q)$. We have

$$\begin{aligned}
A_{\varepsilon_i}\widetilde{f}(z) - A_{\varepsilon_{i+1}}\widetilde{f}(z) \;&=\; \frac{1}{c_d\varepsilon_i^d}\int_{B(z,\varepsilon_i)}\widetilde{f} - \frac{1}{c_d\varepsilon_{i+1}^d}\int_{B(z,\varepsilon_{i+1})}\widetilde{f}\\
&=\; \left(\frac{1}{c_d\varepsilon_i^d} - \frac{1}{c_d\varepsilon_{i+1}^d}\right)\int_{B(z,\varepsilon_i)}\widetilde{f} + \frac{1}{c_d\varepsilon_{i+1}^d}\left(\int_{B(z,\varepsilon_i)}\widetilde{f} - \int_{B(z,\varepsilon_{i+1})}\widetilde{f}\right)\\
&=\; \left(\frac{1}{c_d\varepsilon_i^d} - \frac{1}{c_d\varepsilon_{i+1}^d}\right)\int_{B(z,\varepsilon_i)}\widetilde{f} + \frac{1}{c_d\varepsilon_{i+1}^d}\left(-\int_{B(z,\varepsilon_{i+1})\setminus B(z,\varepsilon_i)}\widetilde{f}\right) \;=:\; II_1^i + II_2^i.
\end{aligned}$$

The term II_2^i is easy: we get

$$\begin{aligned}
\sum_i |II_2^i| \;&=\; \sum_i \frac{1}{\ell(Q)^d}\int_{B(z,\varepsilon_{i+1})\setminus B(z,\varepsilon_i)}|\widetilde{f}|\\
&=\; \frac{1}{\ell(Q)^d}\sum_i\int_{\varepsilon_i<|z-y|<\varepsilon_{i+1}}|f(y)|\,\mathrm{d}y\\
&\le\; \frac{1}{\ell(Q)^d}\int_{\ell(Q)<|z-y|<2\sqrt{d}\ell(Q)}|f(y)|\,\mathrm{d}y\\
&\le\; \frac{c_d}{|B(z,2\sqrt{d}\ell(Q))}\int_{B(z,2\sqrt{d}\ell(Q))}|f(y)|\,\mathrm{d}y \;\le\; c_dMf(x).
\end{aligned}$$

We need to be a little bit more careful with the term II_1^i. First, we notice that

$$\begin{aligned}
\sum_i |II_1^i| \;&\le\; \sum_i\left(1-\frac{\varepsilon_i^d}{\varepsilon_{i+1}^d}\right)\frac{1}{\varepsilon_i^d}\int_{B(z,\varepsilon_i)}|f|\\
&\le\; \sum_i\left(1-\frac{\varepsilon_i^d}{\varepsilon_{i+1}^d}\right)\frac{c_d}{|B(z,2\sqrt{d}\ell(Q))|}\int_{B(z,2\sqrt{d}\ell(Q))}|f|\\
&\le\; c_dMf(x)\sum_i\left(1-\frac{\varepsilon_i^d}{\varepsilon_{i+1}^d}\right).
\end{aligned}$$

Let us denote $g(x) = x^d$. For the numbers $1 - \frac{\varepsilon_i^d}{\varepsilon_{i+1}^d} = 1^d - \left(\frac{\varepsilon_i}{\varepsilon_{i+1}}\right)^d$ we use the mean value theorem: for every i there exists a number $\xi_i \in (0,1)$ such that

$$1^d - \left(\tfrac{\varepsilon_i}{\varepsilon_{i+1}}\right)^d \;=\; g'(\xi_i)\left(1 - \tfrac{\varepsilon_i}{\varepsilon_{i+1}}\right) \;=\; d\xi_i^{d-1}\left(1 - \tfrac{\varepsilon_i}{\varepsilon_{i+1}}\right) \;\le\; d\left(1 - \tfrac{\varepsilon_i}{\varepsilon_{i+1}}\right).$$

In particular, we get

$$\begin{aligned}
\sum_i \left(1 - \frac{\varepsilon_i^d}{\varepsilon_{i+1}^d}\right) \;\le\; d\sum_i \left(1 - \frac{\varepsilon_i}{\varepsilon_{i+1}}\right) \;&=\; d\sum_i \frac{\varepsilon_{i+1} - \varepsilon_i}{\varepsilon_{i+1}} \\
&\le\; \frac{d}{\ell(Q)} \sum_i (\varepsilon_{i+1} - \varepsilon_i) \\
&=\; \frac{d}{\ell(Q)} (\varepsilon_N - \varepsilon_0) \\
&\le\; \frac{d}{\ell(Q)} (2\sqrt{d}\ell(Q) - \ell(Q)) \;=\; c_d.
\end{aligned}$$

Thus, we get $II \le c_d Mf(x)$ and the bound (4) follows. By Theorem 1.8.3, this bound is enough for us to conclude that $M_{V^r A}\colon L^1 \to L^{1,\infty}$. $\square$

Exercise 1.8.18 Consider the standard dyadic intervals $\mathscr{D}$ of $\mathbb{R}$, and define the dyadic analogue of the averaging operators A_ε by $E_j f(x) := \langle f \rangle_{Q_j(x)}$, where $Q_j(x)$ is the unique dyadic cube of side-length 2^{-j} that contains x. The corresponding variation operator is $V^r E f := \sup \left(\sum_i |E_{j_i} f - E_{j_1} f|^r\right)^{1/r}$, where the supremum is over all increasing increasing sequences j_i.

Define the L^∞-normalised Haar functions $h_I^\infty := 1_{I_\ell} - 1_{I_r}$, where $I_{\ell/r}$ is the left/right half of I, and the Rademacher functions $r_j := \sum_{I \in \mathscr{D}_j[0,1)} h_I^\infty$, where $\mathscr{D}_j[0,1) = \{I \in \mathscr{D} \colon I \subseteq [0,1), \ell(I) = 2^{-j}\}$. Check that the functions $(r_i)_{i=0}^\infty$ are orthonormal: $\int r_i r_j = \delta_{ij}$ (=: 1 if $i = j$, and =: 0 else). Check that $E_j r_i = r_i$ if $j > i$ and $E_j r_i = 0$ if $j \le i$. Then consider a function of the form $f = \sum_{i=0}^\infty a_i r_i$. Check that, pointwise on $[0,1)$, we have $V^r E f \ge \left(\sum_{i=0}^\infty |a_i|^r\right)^{1/r}$, while $\|f\|_{L^1} \le \|f\|_{L^2} = \left(\sum_{i=0}^\infty |a_i|^2\right)^{1/2}$. Conclude with a suitable choice of $(a_i)_{i=0}^\infty$ that $V^r E \colon L^1 \not\to L^{1,\infty}$ if $r < 2$.

Solution. Since we have to check several small claims, we break the solution into four parts for clarity.

1) *Orthogonality of the functions r_i.*

First, we notice that for any $I, J \in \mathscr{D}([0,1))$ such that $\ell(I) \le \ell(J)$ we have

$$h_I^\infty h_J^\infty = \begin{cases} 1_J, & \text{if } I = J \\ 0, & \text{if } I \cap J = \emptyset \\ h_I^\infty, & \text{if } I \subseteq J_\ell \\ -h_I^\infty, & \text{if } I \subseteq J_r \end{cases} .$$

Thus, for $i \ge j$ we get

$$\begin{aligned} r_i r_j &= \left(\sum_{I \in \mathscr{D}_i[0,1)} h_I^\infty\right)\left(\sum_{J \in \mathscr{D}_j[0,1)} h_J^\infty\right) \\ &= \sum_{J \in \mathscr{D}_j[0,1)} \sum_{\substack{I \in \mathscr{D}_i[0,1) \\ I = J}} 1_J + \sum_{J \in \mathscr{D}_j[0,1)} \sum_{\substack{I \in \mathscr{D}_i[0,1) \\ I \subseteq J_\ell}} h_I^\infty + \sum_{J \in \mathscr{D}_j[0,1)} \sum_{\substack{I \in \mathscr{D}_i[0,1) \\ I \subseteq J_r}} - h_I^\infty \\ &=: I + II + III \\ &= \begin{cases} I, & \text{if } i = j \\ II + III, & \text{if } i < j \end{cases} \\ &= \begin{cases} 1_{[0,1)}, & \text{if } i = j \\ II + III, & \text{if } i < j \end{cases} . \end{aligned}$$

Since the supports of the functions $h_{I_1}^\infty$ and $h_{I_2}^\infty$ are disjoint if $I_1, I_2 \in \mathscr{D}_i[0,1)$ and we have $\int h_I^\infty = 0$ for any $I \in \mathscr{D}[0,1)$, we get

$$\begin{aligned} \int r_i r_j &= \begin{cases} \int I_{[0,1)}, & \text{if } i = j \\ \sum_{J \in \mathscr{D}_j[0,1)} \sum_{\substack{I \in \mathscr{D}_i[0,1) \\ I \subseteq J_\ell}} \int h_I^\infty + \sum_{J \in \mathscr{D}_j[0,1)} \sum_{\substack{I \in \mathscr{D}_i[0,1) \\ I \subseteq J_r}} \int -h_I^\infty, & \text{if } i < j \end{cases} \\ &= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i < j \end{cases} \\ &= \delta_{ij}. \end{aligned}$$

2) *The function $E_j r_i$.*

Suppose that $x \in J \in \mathscr{D}_j[0,1)$.

- Suppose that $i \ge j$. Now we have $\int_J h_I^\infty = 0$ for every $I \in \mathscr{D}_i[0,1)$ and thus,

$$E_j r_i(x) = \langle r_i \rangle_J = \sum_{\substack{I \in \mathscr{D}_i[0,1) \\ I \subseteq J}} \langle h_I^\infty \rangle_J = 0$$

- Suppose that $i < j$. Now there exists exactly one $I \in \mathscr{D}_i[0,1)$ such that $I \cap J \ne \emptyset$. Since $i < j$, we know that either $J \subset I_\ell$ or $J \subset I_r$. Thus,

$$E_j r_i(x) = \langle r_i \rangle_J = \langle h_I^\infty \rangle_J = \begin{cases} \langle 1 \rangle_J, & \text{if } J \subset I_\ell \\ \langle -1 \rangle_J, & \text{if } J \subset I_r \end{cases} = \begin{cases} 1, & \text{if } J \subset I_\ell \\ -1, & \text{if } J \subset I_r \end{cases} = h_I^\infty(x) = r_i(x).$$

3) *Estimates for functions of the type* $f = \sum_{i=0}^{\infty} a_i r_i$.

Let us notice that for any $j \geq 0$ and $x \in [0,1)$ the part 2) gives us

$$|E_j f(x) - E_{j+1} f(x)| \;=\; \left| \sum_{i=0}^{j-1} a_i r_i(x) - \sum_{i=0}^{j} a_i r_i(x) \right| \;=\; |a_j r_j(x)| \;=\; |a_j|.$$

In particular, we get

$$V^r E f(x) \;\geq\; \left(\sum_{j=0}^{\infty} |E_j f(x) - E_{j+1} f(x)|^r \right)^{1/r} \;=\; \left(\sum_{j=0}^{\infty} |a_j|^r \right)^{1/r} \tag{5}$$

for every $x \in [0,1)$. Also, by the orthogonality (O) of the Rademacher functions, we have

$$\begin{aligned} \|f\|_{L^2} \;=\; \lim_{k\to\infty} \left(\int \left(\sum_{i=0}^{k} a_i r_i(x) \right)^2 \mathrm{d}x \right)^{1/2} &\;=\; \lim_{k\to\infty} \left(\int \sum_{i,j=0}^{k} a_i a_j r_i(x) r_j(x) \,\mathrm{d}x \right)^{1/2} \\ &\;=\; \lim_{k\to\infty} \left(\sum_{i,j=0}^{k} a_i a_j \int r_i(x) r_j(x) \,\mathrm{d}x \right)^{1/2} \\ &\overset{(O)}{=} \lim_{k\to\infty} \left(\sum_{i=0}^{k} a_i^2 \right)^{1/2} \;=\; \left(\sum_{i=0}^{\infty} |a_i|^2 \right)^{1/2}, \end{aligned} \tag{6}$$

and Hölder's inequality gives us

$$\|f\|_{L^1([0,1))} \;\leq\; \|f\|_{L^2([0,1))} \|1_{[0,1)}\|_{L^2([0,1))} \;\leq\; \|f\|_{L^2([0,1))}. \tag{7}$$

4) *A counterexample.*

Suppose that $\varepsilon > 0$ and $r < 2$. We set

$$a_0^\varepsilon := 1, \qquad a_n^\varepsilon := \frac{1}{n^{1/r+\varepsilon}}, \qquad f_\varepsilon := \sum_{n=0}^{\infty} a_n^\varepsilon r_n.$$

Since we have $2(1/r + \varepsilon) > 1$, we know that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ and thus, $\|f_\varepsilon\|_{L^1} \leq \|f_\varepsilon\|_{L^2} < \infty$ and $f_\varepsilon \in L^1$. Also, since $1/r - 1/2 > 0$, we get

$$\begin{aligned} V^r E f_\varepsilon(x) \overset{(5)}{\geq} \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^r \right)^{1/r} &\;=\; \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^r \right)^{1/r-1/2} \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^r \right)^{1/2} \\ &\;>\; \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^r \right)^{1/r-1/2} \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^2 \right)^{1/2} \\ &\overset{(6)}{=} \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^r \right)^{1/r-1/2} \|f_\varepsilon\|_{L^2} \\ &\overset{(7)}{\geq} \left(\sum_{n=0}^{\infty} |a_n^\varepsilon|^r \right)^{1/r-1/2} \|f_\varepsilon\|_{L^1} \\ &\;:=\; A_{r,\varepsilon} \|f_\varepsilon\|_{L^1}. \end{aligned}$$

In particular, we have

$$\begin{aligned} \|V^r E f_\varepsilon\|_{L^{1,\infty}} &\;=\; \sup_{t>0} t \cdot |\{x \in [0,1) \colon V^r E f_\varepsilon(x) > t\}| \\ &\;\geq\; A_{r,\varepsilon} \|f_\varepsilon\|_{L^1} \cdot |\{x \in [0,1) \colon V^r E f_\varepsilon(x) > A_{r,\varepsilon} \|f_\varepsilon\|_{L^1}\}| \\ &\;=\; A_{r,\varepsilon} \|f_\varepsilon\|_{L^1}. \end{aligned}$$

Thus, $\|V^r E\|_{L^1 \to L^{1,\infty}} \geq A_{r,\varepsilon}$. Since $A_{r,\varepsilon} \nearrow \infty$ as $\varepsilon \searrow 0$, we have $\|V^r E\|_{L^1 \to L^{1,\infty}} = \infty$. Hence, if $r < 2$, then $V^r E \colon L^1 \not\to L^{1,\infty}$.

□