Dyadic analysis and weights, Spring 2017
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Solutions to the exercise set 2 (6 pages)

Exercise 1.6.6. Check that there are constants c, c' such that every modulus of continuity ω satisfies

$$c\|\omega\|_{\text{Dini}} \leq \sum_{k=1}^{\infty} \omega(2^{-k}) \leq c'\|\omega\|_{\text{Dini}}.$$

Solution. We first notice that for all $m \geq 1$ we have

$$\int_{2^{-m}}^{2^{-m+1}} \frac{1}{t}\,\mathrm{d}t \;\leq\; \int_{2^{-m}}^{2^{-m+1}} \frac{1}{2^{-m}}\,\mathrm{d}t \;=\; \frac{1}{2^{-m}}\left(2^{-m+1} - 2^{-m}\right) \;=\; 1 \tag{1}$$

and

$$\int_{2^{-m}}^{2^{-m+1}} \frac{1}{t}\,\mathrm{d}t \;\geq\; \int_{2^{-m}}^{2^{-m+1}} \frac{1}{2^{-m+1}}\,\mathrm{d}t \;=\; \frac{1}{2^{-m+1}}\left(2^{-m+1} - 2^{-m}\right) \;=\; \frac{1}{2}. \tag{2}$$

Thus, since the function ω is sub-additive (SA) and increasing (In.), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \omega(2^{-k}) \;=\; \frac{1}{3}\sum_{k=1}^{\infty} 3\omega(2^{-k}) \;\geq\; \frac{1}{3}\left(\sum_{k=1}^{\infty}\omega(2^{-k}) + \omega(\tfrac{1}{2}) + \omega(\tfrac{1}{2})\right) &\overset{\text{(SA)}}{\geq} \frac{1}{3}\left(\sum_{k=1}^{\infty}\omega(2^{-k}) + \omega(\tfrac{1}{2} + \tfrac{1}{2})\right) \\
&= \frac{1}{3}\sum_{k=0}^{\infty}\omega(2^{-k}) \\
&\overset{(1)}{\geq} \frac{1}{3}\sum_{k=0}^{\infty}\omega(2^{-k}) \int_{2^{-k+1}}^{2^{-k}} \frac{1}{t}\,\mathrm{d}t \\
&\overset{\text{(In.)}}{\geq} \frac{1}{3}\sum_{k=0}^{\infty}\int_{2^{-k+1}}^{2^{-k}} \omega(t)\,\frac{\mathrm{d}t}{t} \\
&= \frac{1}{3}\|\omega\|_{\text{Dini}}
\end{aligned}$$

and

$$\sum_{k=1}^{\infty} \omega(2^{-k}) \;\overset{(2)}{\leq}\; 2\sum_{k=1}^{\infty}\omega(2^{-k})\int_{2^{-k}}^{2^{-k+1}} \frac{1}{t}\,\mathrm{d}t \;\overset{\text{(In.)}}{\leq}\; 2\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}} \omega(t)\,\frac{\mathrm{d}t}{t} \;=\; 2\|\omega\|_{\text{Dini}}.$$

Thus, we may simply choose $c = \frac{1}{3}$ and $c' = 2$. □

Exercise 1.6.7. Consider Lerner's maximal operator M_T, when $T = M$, the Hardy-Littlewood maximal operator, and show that $M_M f \le c_d M f$.

Solution. In this solution, Q and P are cubes. Recall that Lerner's maximal operator M_T was defined pointwise as

$$M_T f(x) = \sup_{Q \ni x} \sup_{y \in Q} |T(1_{(3Q)^c} f)(y)|.$$

Suppose that $x, y \in Q$. Then, if $y \in P$ and $P \cap (3Q)^c \neq \emptyset$, we have $\ell(P) \ge \ell(Q)$. In particular, since $\|x - y\|_\infty \le \ell(Q)$, we have $x \in 3P$ for every such cube. We also note that $|3P| = 3^d |P|$. This gives us

$$\begin{aligned}
M_T f(x) &= \sup_{Q \ni x} \sup_{y \in Q} |M(1_{(3Q)^c} f)(y)| \\
&= \sup_{Q \ni x} \sup_{y \in Q} \sup_{P \ni y} \frac{1}{|P|} \int_P 1_{(3Q)^c}(z) |f(z)| \, \mathrm{d}z \\
&= \sup_{Q \ni x} \sup_{y \in Q} \sup_{\substack{P \ni y \\ \ell(P) \ge \ell(Q)}} \frac{1}{|P|} \int_P 1_{(3Q)^c}(z) |f(z)| \, \mathrm{d}z \\
&\le 3^d \sup_{Q \ni x} \sup_{y \in Q} \sup_{\substack{P \ni y \\ \ell(P) \ge \ell(Q)}} \frac{1}{|3P|} \int_{3P} 1_{(3Q)^c}(z) |f(z)| \, \mathrm{d}z \\
&\le 3^d \sup_{Q \ni x} \sup_{y \in Q} \sup_{P \ni x} \frac{1}{|P|} \int_P 1_{(3Q)^c}(z) |f(z)| \, \mathrm{d}z \\
&\le 3^d \sup_{Q \ni x} \sup_{y \in Q} \sup_{P \ni x} \frac{1}{|P|} \int_P |f(z)| \, \mathrm{d}z \\
&\le 3^d \sup_{Q \ni x} \sup_{y \in Q} Mf(x) \\
&= 3^d Mf(x),
\end{aligned}$$

which is what we wanted. □

Exercise 1.6.8. Prove the analogue of Lemma 1.6.2 for the maximal truncated Calderón-Zygmund operator $T_\sharp$ in place of the linear Calderón-Zygmund operator T, i.e. prove a pointwise bound for $M_{T_\sharp}$ which allows to conclude the $L^1 \to L^{1,\infty}$ boundedness of this operator, and hence the A_2 theorem for $T_\sharp$.

Solution. Since

$$M_{T_\sharp} f(x) \;=\; \sup_{Q \ni x} \sup_{z \in Q} \left| T_\sharp (1_{(3Q)^c} f)(z) \right| \;=\; \sup_{Q \ni x} \sup_{z \in Q} \sup_{\varepsilon > 0} \left| T_\varepsilon (1_{(3Q)^c} f)(z) \right|,$$

it suffices to find a (ε, Q, z)-independent pointwise bound for $\left| T_\varepsilon (1_{(3Q)^c} f)(z) \right|$. We only need to slightly modify the proof of Lemma 1.6.2 to find this bound.

Let us fix arbitrary $Q \ni x$, $z \in Q$ and $\varepsilon > 0$. We set $\varepsilon_{\max} := \max\{\varepsilon, 2\sqrt{d}\ell(Q)\}$ and $B_x^\varepsilon := B(x, \varepsilon_{\max})$. We have

$$T_\varepsilon (1_{(3Q)^c} f)(z) \;=\; T(1_{B(x,\varepsilon)^c} 1_{(3Q)^c} f)(z) \;=\; T(1_{(3Q \cup B(x,\varepsilon))^c} f)(z).$$

and $3Q \cup B(x, \varepsilon) \subset B_x^\varepsilon$. We notice that

$$T_\varepsilon (1_{(3Q)^c} f)(z) \;=\; T_\varepsilon (1_{(3Q)^c} f)(z) - T_{\varepsilon_{\max}} f(x) + T_{\varepsilon_{\max}} f(x) \;\le\; T_\varepsilon (1_{(3Q)^c} f)(z) - T_{\varepsilon_{\max}} f(x) + T_\sharp f(x).$$

We then write

$$\begin{aligned} T_\varepsilon (1_{(3Q)^c} f)(z) - T_{\varepsilon_{\max}} f(x) \;&=\; \int_{(3Q \cup B(x,\varepsilon))^c} K(z,y) f(y)\,\mathrm{d}y - \int_{|y-x| > \varepsilon_{\max}} K(x,y) f(y)\,\mathrm{d}y \\ &=\; \int_{|y-x| > \varepsilon_{\max}} \left(K(z,y) - K(x,y) \right) f(y)\,\mathrm{d}y + \int_{(3Q \cup B(x,\varepsilon))^c \cap B_x^\varepsilon} K(z,y) f(y)\,\mathrm{d}y \\ &=: \; I + II. \end{aligned}$$

The term I is very easy since we can simply forget the ε-dependency by using a crude estimate: we have

$$|I| \;\le\; \int_{|y-x| > 2\sqrt{d}\ell(Q)} |K(z,y) - K(x,y)|\, |f(y)|\,\mathrm{d}y \;\le\; \int_{|y-x| > 2\sqrt{d}\ell(Q)} \omega\left(\frac{|z-y|}{|x-y|} \right) \frac{1}{|x-y|^d} |f(y)|\,\mathrm{d}y.$$

Now we can simply proceed just as in the proof of Lemma 1.6.3 and get

$$|I| \le \|\omega\|_{\mathrm{Dini}} c_d M f(x).$$

We can use the ideas from the proof of Lemma 1.6.3 also for the term II. We have $z \in Q$ and $y \in (3Q \cup B(x,\varepsilon))^c$. In particular, $|y - z| \ge \max\{\ell(Q), \varepsilon - \ell(Q)\} \ge c_d' \varepsilon_{\max}$ for a dimensional constant $c_d' > 0$. Thus,

$$\begin{aligned} |II| \;\le\; \int_{(3Q \cup B(x,\varepsilon))^c \cap B_x^\varepsilon} |K(z,y)| |f(y)|\,\mathrm{d}y \;&\le\; \int_{(3Q \cup B(x,\varepsilon))^c \cap B_x^\varepsilon} \frac{c_K}{|y-z|^d} |f(y)|\,\mathrm{d}y \\ &\le\; c_K c_d'' \int_{B_x^\varepsilon} \frac{1}{\varepsilon_{\max}^d} |f(y)|\,\mathrm{d}y \\ &=\; c_K c_d ⨍_{B_x^\varepsilon} |f(y)|\,\mathrm{d}y \\ &\le\; c_K c_d M f(x). \end{aligned}$$

Hence, we end up with a pointwise bound

$$M_{T_\sharp} f(x) \;\le\; T_\sharp f(x) + c_d \left(c_K + \|\omega\|_{\mathrm{Dini}} \right) M f(x)$$

and the A_2 theorem for $T_\sharp$ follows immediately. $\square$

Exercise 1.6.9. Consider again the Hilbert transform H from Exercise 1.3.5. Taking for granted that $H\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded, check that H is a Calderón-Zygmund operator with modulus of continuity of the form $\omega(t) = ct$ for some constant c. Conclude from the previous results that

$$\|H\|_{L^2(w)\to L^2(w)} \le c[w]_{A_2}, \qquad w \in A_2, \tag{3}$$

and argue by extrapolation (without a concrete example) that this dependence on $[w]_{A_2}$ is optimal.

Solution. Recall that the Hilbert transform is defined pointwise as the operator H,

$$Hf(x) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{f(y)}{x-y}\mathrm{d}y.$$

We claim that H is an Calderón-Zygmund operator with a kernel K,

$$K(x,y) = \frac{1}{x-y}$$

and modulus of continuity ω, $\omega(t) = 4t$.

(1) Suppose that f is a function such that Hf is well-defined almost everywhere and let $x \notin \operatorname{supp} f$. Then, by definition, there exists $\varepsilon_0 > 0$ such that $f|_{(x-\varepsilon_0,x+\varepsilon_0)} \equiv 0$ (*). Thus, we get

$$\begin{aligned} Hf(x) &= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{f(y)}{x-y}\mathrm{d}y \overset{(*)}{=} \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{f(y)1_{(x-\varepsilon_0,x+\varepsilon_0)^c}(y)}{x-y}\mathrm{d}y \\ &= \left(\int_{-\infty}^{x-\varepsilon_0} + \int_{x+\varepsilon_0}^{\infty}\right)\frac{f(y)1_{(x-\varepsilon_0,x+\varepsilon_0)^c}(y)}{x-y}\mathrm{d}y \\ &= \int_{\mathbb{R}}\frac{f(y)1_{(x-\varepsilon_0,x+\varepsilon_0)^c}(y)}{x-y}\mathrm{d}y \\ &\overset{(*)}{=} \int_{\mathbb{R}} K(x,y)f(y)\,\mathrm{d}y. \end{aligned}$$

(2) The size estimate of the kernel is satisfied trivially with $C_K = 1$.

(3) Suppose that $|x-y| > 2|x-x'|$. This gives us

$$|x-x'| + |x'-y| \;\ge\; |x-y| \;>\; 2|x-x'| \implies |x'-y| > |x-x'|. \tag{4}$$

Thus,

$$\begin{aligned} |K(x,y) - K(x',y)| &= \left|\frac{1}{x-y} - \frac{1}{x'-y}\right| = \frac{|x-x'|}{|x-y|}\frac{1}{|x'-y|} \\ &= \frac{|x-x'|}{|x-y|}\frac{|x-y|}{|x'-y|}\frac{1}{|x-y|} \\ &\le \frac{|x-x'|}{|x-y|}\left(\frac{|x-x'|+|x'-y|}{|x'-y|}\right)\frac{1}{|x-y|} \\ &\overset{(4)}{\le} 2\frac{|x-x'|}{|x-y|}\frac{1}{|x-y|}. \end{aligned}$$

Since we have $K(x,y) = -K(y,x)$, we get $|K(x,y)-K(x',y)| = |K(y,x)-K(y,x')|$. In particular,

$$|K(x,y)-K(x',y)| + |K(y,x)-K(y,x')| \;=\; 2|K(x,y)-K(x',y)| \;\le\; 4\frac{|x-x'|}{|x-y|}\frac{1}{|x-y|}.$$

Hence, the Hilbert transform is a Calderón-Zygmund operator. In particular, by Theorem 1.6.3, we have

$$\|H\|_{L^2(w)\to L^2(w)} \;\le\; c_d c_H [w]_{A_2}, \qquad w \in A_2.$$

Recall that by Exercise 1.3.5 we have $\|H\|_{L^p\to L^p} \ge cp$ for every $p \in [2,\infty)$. Also, by Lemma 1.3.3, we have $A_1 \subset A_2$ and $[w]_{A_2} \le [w]_{A_1}$. Thus, the bound (3) is optimal with respect to $[w]_{A_2}$ by the Fefferman-Pipher theorem (Theorem 1.3.2): if we had $\|H\|_{L^2(w)\to L^2(w)} \le c_d c_H [w]_{A_2}^{\alpha}$ for some $\alpha < 1$, we would get $\|H\|_{L^p\to L^p} \le c' p^{\alpha}$ for all $p \in [2,\infty)$ which is not true for large p by Exercise 1.3.5. □

Exercise 1.6.11. Show the optimality of (3) be working out the following concrete example (without using extrapolation): Cosider the weight $w(x) = |x|^{\alpha}$, and the function $f(x) = |x|^{-\alpha}1_{(-1,0)}(x)$ and estimate the quantities $[w]_{A_2}$, $\|f\|_{L^2(w)}$ and $\|Hf\|_{L^2(w)}$.

Solution. Let us start by showing[1] that if $w(x) = |x|^{-\beta}$, then $w \in A_2(\mathbb{R})$ if and only if $-1 < \beta < 1$, and for these β we have

$$[w]_{A_2} \lesssim \frac{1}{1-\beta^2}. \tag{5}$$

The necessity of the condition $-1 < \beta < 1$ is obvious since we can simply consider cubes of the form $(\varepsilon, 1)$ and take the limit $\varepsilon \searrow 0$ to see that $[w]_{A_2}$ is not finite if $\beta \notin (-1,1)$. Thus, we only need to show that if $\beta \in (-1,1)$, then (5) holds. We prove this in two parts.

1) Suppose that $|a| \geq 2r$. Then, since $|a| + r \geq |a \pm r|$ and $|a| - r \leq |a \pm r|$, we get

$$\frac{1}{2r}\int_{a-r}^{a+r} |x|^{\beta}\,\mathrm{d}x \;\leq\; \begin{cases} (|a|+r)^{\beta}, & \text{if } \beta > 0 \\ (|a|-r)^{\beta}, & \text{if } \beta \leq 0 \end{cases} \;\leq\; \begin{cases} (2|a|)^{\beta}, & \text{if } \beta > 0 \\ \left(\frac{a}{2}\right)^{\beta}, & \text{if } \beta \leq 0 \end{cases} \;=\; 2^{|\beta|}|a|^{\beta}.$$

In particular,

$$\langle w\rangle_{(a-r,a+r)}\langle w^{-1}\rangle_{(a-r,a+r)} \;\leq\; 2^{|\beta|}|a|^{\beta}\cdot 2^{|\beta|}|a|^{-\beta} \;=\; 4^{|\beta|} \;\leq\; 4 \;\lesssim\; \frac{1}{1-\beta^2}.$$

2) Suppose then that $|a| < 2r$. Then $(a-r, a+r) \subset (-3r, 3r)$ and we get

$$\frac{1}{2r}\int_{a-r}^{a+r} |x|^{\beta}\,\mathrm{d}x \;\leq\; \frac{1}{2r}\int_{-3r}^{3r} |x|^{\beta}\,\mathrm{d}x \;=\; \frac{1}{r}\int_{0}^{3r} x^{\beta}\,\mathrm{d}x \;=\; \frac{1}{r}\frac{(3r)^{\beta+1}}{\beta+1} \;=\; \frac{3^{\beta+1}}{\beta+1}r^{\beta}.$$

Furthermore,

$$\langle w\rangle_{(a-r,a+r)}\langle w^{-1}\rangle_{(a-r,a+r)} \;\leq\; \frac{3^{\beta+1}}{\beta+1}r^{\beta}\cdot\frac{3^{-\beta+1}}{-\beta+1}r^{-\beta} \;=\; \frac{3^2}{(1+\beta)(1-\beta)} \;\lesssim\; \frac{1}{1-\beta^2}.$$

Combining parts 1) and 2) gives us (5). Let us then prove the optimality of (3). Suppose that $\alpha \in (0,1)$. We first notice that for $x \in (0,1)$ we have

$$|Hf(x)| \;=\; \int_{-1}^{0}\frac{-y^{-\alpha}}{x-y}\mathrm{d}y \;\geq\; \int_{-x}^{0}\frac{-y^{-\alpha}}{x-y}\mathrm{d}y \;\geq\; \frac{1}{2x}\int_{-x}^{0} -y^{-\alpha}\,\mathrm{d}y \;=\; \frac{1}{2x}\cdot\frac{1}{1-\alpha}x^{-\alpha+1} \;=\; \frac{1}{2}\cdot\frac{1}{1-\alpha}|x|^{-\alpha}. \tag{6}$$

Let us then consider the weight w, $w(x) = |x|^{\alpha^{1/2}}$. By the previous part of the solution, we know that $w \in A_2(\mathbb{R})$. Since the weight w is an even function, we get

$$\begin{aligned}
\|Hf\|_{L^2(w)} \;\geq\; \left(\int_0^1 |Hf(x)|^2\cdot w(x)\,\mathrm{d}x\right)^{1/2} &\overset{(6)}{\geq} \frac{1}{2}\cdot\frac{1}{1-\alpha}\left(\int_0^1 |x|^{-2\alpha}\cdot w(x)\,\mathrm{d}x\right)^{1/2} \\
&= \frac{1}{2}\cdot\frac{1}{1-\alpha}\left(\int_{-1}^0 |x|^{-2\alpha}\cdot w(x)\,\mathrm{d}x\right)^{1/2} \\
&= \frac{1}{2}\cdot\frac{1}{1-(\alpha^{1/2})^2}\|f\|_{L^2(w)} \\
&\overset{(5)}{\gtrsim} [w]_{A_2}\|f\|_{L^2(w)}.
\end{aligned}$$

This concludes the proof. ☐

[1] This results holds more generally in the following form. Suppose that $w(x) = |x|^{\beta}$. Then, for $p > 1$, $w \in A_p(\mathbb{R}^d)$ if and only if $-d < \beta < d(p-1)$. In this range we have $[w]_{A_p} \eqsim_{p,d} \frac{1}{(d+\beta)(d(p-1)-\alpha)^{p-1}}$.

Exercise 1.7.4. Suppose that we did the proof of Theorem 1.7.2 only with $\alpha = 1$, leading to the bound

$$\|T\|_{L^1\to L^{1,\infty}} \leq c_d \left(\|T\|_{L^2\to L^2}^2 + 1 + \|\omega\|_{\mathrm{Dini}}\right). \tag{7}$$

Apply this to the operator αT in place of T, where $\alpha > 0$ is a constant, and see how the different quantities depend on α to deduce

$$|\{|Tf| > \lambda\}| \leq \frac{c_d}{\lambda}\|f\|_{L^1}\left(\alpha\|T\|_{L^2\to L^2}^2 + \frac{1}{\alpha} + \|\omega\|_{\mathrm{Dini}}\right)$$

and thus the statement of Theorem 1.7.2 by this alternative route.

Solution. Suppose that T is a Calderón-Zygmund operator with kernel K and modulus of continuity ω and let $\alpha > 0$. Directly from the definition it follows that αT is also a Calderón-Zygmund operator with kernel αK and modulus of continuity $\alpha\omega$. We also get

$$\begin{aligned}
\|\alpha T\|_{L^2\to L^2}^2 &= \left(\inf\{K \geq 0\colon \|(\alpha T)f\|_{L^2} \leq K\|f\|_{L^2}, f \in L^2\}\right)^2 \\
&= \left(\inf\{K \geq 0\colon \alpha\|Tf\|_{L^2} \leq K\|f\|_{L^2}, f \in L^2\}\right)^2 \\
&= \left(\alpha\inf\{K \geq 0\colon \|Tf\|_{L^2} \leq K\|f\|_{L^2}, f \in L^2\}\right)^2 \\
&= \alpha^2\|T\|_{L^2\to L^2}
\end{aligned}$$

and

$$\|\alpha\omega\|_{\mathrm{Dini}} = \int_0^1 \alpha\omega(t)\,\frac{\mathrm{d}t}{t} = \alpha\int_0^1 \omega(t)\,\frac{\mathrm{d}t}{t} = \alpha\|\omega\|_{\mathrm{Dini}}.$$

Suppose that $f \in L^1$. We get

$$\begin{aligned}
|\{|Tf| > \lambda\}| &= |\{|(\alpha T)f| > \alpha\lambda\}| \\
&\overset{(7)}{\leq} \frac{c_d}{\alpha\lambda}\|f\|_{L^1}\left(\|\alpha T\|_{L^2\to L^2}^2 + 1 + \|\alpha\omega\|_{\mathrm{Dini}}\right) \\
&= \frac{c_d}{\lambda}\|f\|_{L^1}\left(\frac{\alpha^2}{\alpha}\|T\|_{L^2\to L^2}^2 + \frac{1}{\alpha} + \frac{\alpha}{\alpha}\|\omega\|_{\mathrm{Dini}}\right) \\
&= \frac{c_d}{\lambda}\|f\|_{L^1}\left(\alpha\|T\|_{L^2\to L^2}^2 + \frac{1}{\alpha} + \|\omega\|_{\mathrm{Dini}}\right),
\end{aligned}$$

which is what we wanted. □