Dyadic analysis and weights, Spring 2017
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Solutions to the exercise set 1 (4 pages)

Exercise 1.2.3 For a given weight w and $p \in (1, \infty)$, find the L^p dual weight σ such that the inequalities $\|Tf\|_{L^p(w)} \leq K\|f\|_{L^p(w)}$ and $\|T(f\sigma)\|_{L^p(w)} \leq K\|f\|_{L^p(\sigma)}$ (for all f that make the respective right sides finite) are equivalent.

Solution. We simply notice that if the inequality $\|Tf\|_{L^p(w)} \leq K\|f\|_{L^p(w)}$ holds, we have

$$\|T(f\sigma)\|_{L^p(w)} \leq K\|f\sigma\|_{L^p(\sigma)} = K\left(\int_{\mathbb{R}^d} |f\sigma|^p w\right)^{1/p} = K\left(\int_{\mathbb{R}^d} |f|^p(\sigma^p w)\right)^{1/p} = K\|f\|_{L^p(\sigma^p w)}.$$

Thus, in order for the inequality $\|T(f\sigma)\|_{L^p(w)} \leq K\|f\|_{L^p(\sigma)}$ to hold, we need to have $\sigma^p w = \sigma$. In particular, we get

$$\sigma = w^{-1/(p-1)}.$$

For this choice of σ, it is simple to show that the latter inequality implies the first inequality:

$$\begin{aligned}\|Tf\|_{L^p(w)} = \|T((f\sigma^{-1})\sigma)\|_{L^p(w)} \leq K\|f\sigma^{-1}\|_{L^p(\sigma)} &= K\left(\int_{\mathbb{R}^d} |f\sigma^{-1}|^p \sigma\right)^{1/p} \\ &= K\left(\int_{\mathbb{R}^d} |f|^p \sigma^{1-p}\right)^{1/p} \\ &= K\left(\int_{\mathbb{R}^d} |f|^p w\right)^{1/p} \\ &= K\|f\|_{L^p(w)}.\end{aligned}$$

□

Exercise 1.2.4 Let w be a weight, and consider the operator $f \mapsto 1_Q\langle f\rangle_Q$. Show that the norm of this operator on $L^2(w)$ is $(\langle w\rangle_Q\langle w^{-1}\rangle_Q)^{1/2}$.

Solution. Take $p = 2$ in the proof of the next exercise. □

Exercise 1.2.5 For $p \in (1, \infty)$, find the norm of the operator $f \mapsto 1_Q \langle f \rangle_Q$ on $L^p(w)$.

Solution. Let us denote the operator by T and use the dual weight notation from Exercise 1.2.3: $\sigma = w^{-1/(p-1)}$. We claim that

$$\|T\|_{L^p(w)} = \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'} .$$

For the upper bound, we can simply use Hölder's inequality to see that

$$\begin{aligned}
\|Tf\|_{L^p(w)} = \left(\int_{\mathbb{R}^d} 1_Q \langle f \rangle_Q^p w \right)^{1/p} = \langle f \rangle_Q \left(\int_Q w \right)^{1/p} &= \frac{1}{|Q|} \left(\int_Q |f| \right) w(Q)^{1/p} \\
&= \frac{1}{|Q|^{1/p+1/p'}} \left(\int_Q |f| w^{1/p} \cdot w^{-1/p} \right) w(Q)^{1/p} \\
&\leq \frac{1}{|Q|^{1/p'}} \left(\int_Q |f|^p w \right)^{1/p} \left(\int_Q w^{-p'/p} \right)^{1/p'} \langle w \rangle_Q^{1/p} \\
&\leq \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'} \|f\|_{L^p(w)}.
\end{aligned}$$

Thus, $\|T\|_{L^p(w)} \leq \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'}$.

Let us then prove the lower bound. Since the function σ may not be integrable over Q, we consider the function $\sigma_\varepsilon := (w + \varepsilon)^{-1/(p-1)}$. The function σ_ε is integrable over Q and we have $\sigma_\varepsilon \nearrow \sigma$ pointwise as $\varepsilon \searrow 0$. We then set

$$f_\varepsilon := 1_Q \sigma_\varepsilon \langle \sigma_\varepsilon \rangle_Q^{-1/p} |Q|^{-1/p}.$$

This gives us

$$Tf_\varepsilon \;=\; 1_Q \langle f_\varepsilon \rangle_Q \;=\; 1_Q \langle \sigma_\varepsilon \rangle_Q^{-1/p} |Q|^{-1/p} \langle \sigma_\varepsilon \rangle_Q \;=\; 1_Q \langle \sigma_\varepsilon \rangle_Q^{1/p'} |Q|^{-1/p}.$$

In particular:

$$\|Tf_\varepsilon\|_{L^p(w)} = \left(\int (Tf_\varepsilon)^p w \right)^{1/p} = \langle \sigma_\varepsilon \rangle_Q^{1/p'} \left(|Q|^{-1} \int_Q w \right)^{1/p} = \langle \sigma_\varepsilon \rangle_Q^{1/p'} \langle w \rangle_Q^{1/p}.$$

Thus, taking the limit $\varepsilon \searrow 0$ and using the monotone convergence theorem give us:

- if σ is not integrable over Q, the operator T is not bounded on L^1_{loc};
- if σ is integrable over Q, we have $\|Tf_0\|_{L^p(w)} = \langle \sigma \rangle_Q^{1/p'} \langle w \rangle_Q^{1/p}$.

In the latter case, we also have

$$\|f_0\|_{L^p(w)} = \langle \sigma \rangle_Q^{-1/p} \left(|Q|^{-1} \int_Q \sigma^p \cdot w \right)^{1/p} = \langle \sigma \rangle_Q^{-1/p} \left(|Q|^{-1} \int_Q \sigma \right)^{1/p} = 1,$$

which gives us

$$\|Tf_0\|_{L^p(w)} = \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'} = \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'} \|f_0\|_{L^p(w)}.$$

In particular, $\|T\|_{L^p(w)} \geq \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'}$. □

Exercise 1.3.4 Check that the function $f = 1_{(0,1)}(x)\log x$ satisfies $\|f\|_{L^p} \geq cp$ for $p \in [1,\infty)$.

Solution. Since the function $|f|^p$ is decreasing on the interval $(0,1)$, for every $t \in (0, \frac{1}{e^p})$ we have

$$|f(t)|^p = |\log t|^p \geq |\log \tfrac{1}{e^p}|^p = p^p.$$

In particular, we have

$$\|f\|_{L^p} = \left(\int_0^1 |\log t|^p \mathrm{d}t\right)^{1/p} \geq \left(\int_0^{1/e^p} |\log t|^p \mathrm{d}t\right)^{1/p} \geq \left(\int_0^{1/e^p} p^p \mathrm{d}t\right)^{1/p} = \frac{p}{e}.$$

□

Exercise 1.3.5 Compute the Hilbert transform

$$Hf(x) := \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{f(y)}{x-y}\mathrm{d}y$$

of $f = 1_{(0,1)}$, and deduce that $\|H\|_{L^p\to L^p} \geq cp$ for $p \in [2,\infty)$.

Solution. Let us show that $Hf(0) = -\infty$, $Hf(1) = \infty$ and $Hf(x) = \log|x| - \log|x-1| = \log\left|\frac{x}{1-x}\right|$ for $x \neq 0, 1$.

<u>$x \notin [0,1]$</u> The cases $x < 0$ and $x > 0$ are very similar, so we will only consider the case $x < 0$. If ε is small enough, we have $x-\varepsilon, x+\varepsilon < 0$. Thus,

$$Hf(x) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{1_{(0,1)}(y)}{x-y}\mathrm{d}y = \int_0^1 \frac{1}{x-y}\mathrm{d}y = [-\log|x-y|]_{y=0}^{y=1} = \log|x| - \log|x-1|.$$

<u>$x = 0, 1$</u> We have

$$Hf(0) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{0-\varepsilon} + \int_{0+\varepsilon}^{\infty}\right)\frac{1_{(0,1)}(y)}{-y}\mathrm{d}y = -\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{y}\mathrm{d}y = -\infty.$$
$$Hf(1) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{1_{(0,1)}(y)}{1-y}\mathrm{d}y = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{1-y}\mathrm{d}y = \infty.$$

<u>$x \in (0,1)$</u> If ε is small enough, we have $x-\varepsilon, x+\varepsilon \in (0,1)$. Thus,

$$\begin{aligned}Hf(x) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{1_{(0,1)}(y)}{x-y}\mathrm{d}y &= \lim_{\varepsilon\to 0}\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^{1}\right)\frac{1}{x-y}\mathrm{d}y \\ &= \lim_{\varepsilon\to 0}(\log|x| - \log|x-(x-\varepsilon)| + \log|x-(x+\varepsilon)| - \log|x-1|) \\ &= \log|x| - \log|x-1|.\end{aligned}$$

Thus, we have $Hf(x) = \log\left|\frac{x}{1-x}\right|$ almost everywhere. Since $|Hf|^p$ is decreasing on $(0, 1/(1+e^p))$, we get

$$\begin{aligned}\|Hf\|_{L^p} \geq \left(\int_0^{1/(1+e^p)} |Hf(x)|^p \,\mathrm{d}x\right)^{1/p} &\geq \left(\int_0^{1/(1+e^p)} \left|Hf\left(\frac{1}{1+e^p}\right)\right|^p \mathrm{d}x\right)^{1/p} \\ &= \left(\int_0^{1/(1+e^p)} \left|\log\left(\frac{1}{e^p}\right)\right|^p \mathrm{d}x\right)^{1/p} \\ &= \frac{p}{\sqrt[p]{1+e^p}} \geq \frac{p}{e^2}.\end{aligned}$$

In particular, $\|H\|_{L^p\to L^p} \geq cp$. □

Exercise 1.3.6 Prove an analogue of Theorem 1.3.2 starting from the assumption that $\|T\|_{L^s(w)\to L^s(w)} \leq \phi([w]_{A_1})$ for a fixed $s \neq 2$.

Solution. We prove the following theorem:

Theorem. *Let T be an operator that satisfies*

$$\|T\|_{L^s(w)\to L^s(w)} \leq \phi([w]_{A_1}) \tag{1}$$

for a fixed $s \in [1,\infty)$ and all $w \in A_1$. Then

$$\|T\|_{L^p\to L^p} \leq \sqrt[s]{2}\phi\left(c_d\frac{2p}{s}\right)$$

for $p \in [s,\infty)$. Here c_d is a dimensional constant and we may choose $c_d = 1$ if we replace A_1 by $A_1^{\mathscr{D}}$ in the assumption.

We only need to modify the proof of Theorem 1.3.2 in a couple of parts to prove this theorem. As in the proof of Theorem 1.3.2, we prove both the dyadic and non-dyadic versions of the theorem simultaneously: we get the proof of the dyadic version of the theorem by simply replacing M by $M^{\mathscr{D}}$ and A_1 by $A_1^{\mathscr{D}}$ in the following lines.

We first notice that since $p \geq s$, we have $p/s \geq 1$ and

$$\|Tf\|_{L^p} = \||Tf|^s\|_{L^{p/s}}^{1/s} = \sup\left\{\left(\int_{\mathbb{R}^d}|Tf|^s g\right)^{1/s} : \|g\|_{L^{(p/s)'}} \leq 1\right\}.$$

Let us set $q = (p/s)'$ and apply the Rubio de Francia algorithm to the functions g with the parameter q. This gives us functions $R_q g$ such that

i) $R_q g \geq |g|$,

ii) $\|R_q g\|_{L^q} \leq 2\|g\|_{L^q}$,

iii) $[R_q g]_{A_1} \leq 2\|M\|_{L^q\to L^q}$.

Thus, we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^d}|Tf|^s g\right)^{1/s} &\overset{\text{i)}}{\leq} \left(\int_{\mathbb{R}^d}|Tf|^s R_q g\right)^{1/s} \\
&\overset{(1)}{\leq} \phi([R_q g]_{A_1})\left(\int_{\mathbb{R}^d}|f|^s R_q g\right)^{1/s} \\
&\leq \phi([R_q g]_{A_1})\left(\int_{\mathbb{R}^d}(|f|^s)^{p/s}\right)^{s/sp}\left(\int_{\mathbb{R}^d}(R_q g)^q\right)^{1/sq} \\
&= \phi([R_q g]_{A_1})\|f\|_{L^p}\|R_q g\|_{L^q}^{1/s}
\end{aligned}$$

By the property ii) and the fact that $\|g\|_{L^q} \leq 1$, we get $\|R_q g\|_{L^q}^{1/s} \leq 2^{1/s}\|g\|_q^{1/s} \leq \sqrt[s]{2}$. Since $q' = p/s$, the property iii) and the known L^p-bounds of M give us

$$[R_q g]_{A_1} \leq 2\|M\|_{L^q\to L^q} = \begin{cases} 2q' = \frac{2p}{s}, & \text{in the dyadic case} \\ 2c_d q' = c_d\frac{2p}{s}, & \text{in the non-dyadic case}\end{cases}.$$

□