Degree theory and Branched covers

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Preface

These are revised lecture notes for the course "Degree theory and branched covers" lectured for the first time at the University of Jyväskylä Fall 2015. The purpose of these lecture notes is to introduce the necessary theory for the proof of the Chernavskii–Väisälä's theorem, see [Väi66]; see also Chernavskii [Č64].

Theorem (Väisälä, 1966). *Let $f\colon M \to N$ be a discrete and open map between n-manifolds M and N for $n \geq 3$. Then the branch set B_f of f has topological dimension at most $n-2$ and* $\dim B_f = \dim fB_f = \dim f^{-1}fB_f$.

A discrete and open map[1] is called a *branched cover*. Recall that a map is *discrete* if preimage of a point is a discrete set, and a map is *open* if image of an open set is open. A point $x \in M$ in the domain of a mapping $f\colon M \to N$ is a *branch point* if f is not a local homeomorphism at x. The branch set B_f is the set of all branched points of f.

Väisälä's theorem is fundamental in the theory of these mappings. It yields as a corollary that the branch set does not locally separate the domain of the map; the same holds of course for the image of the branch set. As a corollary we obtain

This fact, on the other hand, shows that a branched cover between manifolds is either *orientation preserving* or *orientation reversing*; some authors assume branched covers to be orientation reversing. In similar vein, Väisälä's theorem justifies the name "branched cover":

Corollary. *Let $f\colon M \to N$ be a branched cover between n-manifolds. Then f is either orientation preserving or reversing.*

In fact, as we will later see, a branched cover between manifolds is locally a completion of a covering map.

The proof of Väisälä's theorem requires a substantial amount of preliminary material. The main argument uses local degree theory of proper maps. In order to define the local degree we discuss first (compactly supported) Alexander–Spanier cohomology, which we use to define (local) orientation

[1] In these notes *map* and *mapping* are synonymous and typically refer a *continuous map*.

and the local index of a brached cover. The classical expositions on this theory are Spanier [Spa66] and Massey [Mas78], which we mainly follow.

The realistic goal of the course is to prove a version of Väislä's theorem stating that B_f and fB_f have no interior and do not locally separate manifolds M and N. The degree arguments follow then from Borel [Bor60] and Church–Hemmingsen [CH60].

Bibliography

[Bor60] Armand Borel. *Seminar on transformation groups.* With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960.

[CH60] Philip T. Church and Erik Hemmingsen. Light open maps on n-manifolds. *Duke Math. J*, 27:527–536, 1960.

[Dug78] James Dugundji. *Topology.* Allyn and Bacon, Inc., Boston, Mass.-London-Sydney, 1978. Reprinting of the 1966 original, Allyn and Bacon Series in Advanced Mathematics.

[Eng78] Ryszard Engelking. *Dimension theory.* North-Holland Publishing Co., Amsterdam-Oxford-New York; PWN—Polish Scientific Publishers, Warsaw, 1978. Translated from the Polish and revised by the author, North-Holland Mathematical Library, 19.

[Mas78] William S. Massey. *Homology and cohomology theory.* Marcel Dekker, Inc., New York-Basel, 1978. An approach based on Alexander-Spanier cochains, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46.

[Spa66] Edwin H. Spanier. *Algebraic topology.* McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[Väi66] Jussi Väisälä. Discrete open mappings on manifolds. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 392:10, 1966.

[Č64] A. V. Černavskiĭ. Finite-to-one open mappings of manifolds. *Mat. Sb. (N.S.)*, 65 (107):357–369, 1964.

Contents

Chapter 1

Alexander–Spanier cohomology

In this chapter we discuss the definition and basic properties of Alexander–Spanier cohomology. The use of Alexander–Spanier cohomology in the proof of Väisälä's theorem stems from the good properties of this cohomology with respect to closed sets. To emphasize this aspect we compare it to the more familiar singular cohomology to highlight the differences.

The compactly supported Alexander–Spanier cohomology $H_c^*(X)$ is the homology of a (co)chain complex $(C_c^k(X), d^k)$ which is a quotient complex of a (co)chain complex $(\Phi_c^k(X), \delta^k)$ of k-functions in X. More precisely, we have a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Phi_c^0(X) & \xrightarrow{\delta^0} & \Phi_c^1(X) & \xrightarrow{\delta^1} & \Phi_c^2(X) & \xrightarrow{\delta^2} & \Phi_c^3(X) & \xrightarrow{\delta^3} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_c^0(X) & \xrightarrow{d^0} & C_c^1(X) & \xrightarrow{d^1} & C_c^2(X) & \xrightarrow{d^2} & C_c^3(X) & \xrightarrow{d^3} & \cdots
\end{array}$$

where the vertical arrows are quotient maps, and $H_c^k(X) = \ker d^k / \operatorname{im} d^{k-1}$

It should be noted that we do not discuss the (non-compactly supported) Alexander–Spanier cohomology $H^*(\cdot)$ at all in these notes and merely refer to Spanier [Spa66] for details.

1.1 Space of k-functions

To emphasize the generality of the theory, we assume at this stage only that X is a topological space. To obtain viable theory, more conditions are added in the later sections (and, in the end, we consider actually open and closed sets in Euclidean spaces).

Definition 1.1.1. For $k \in \mathbb{N}$, a *k-function on X* is a function $X^{k+1} \to$

$\mathbb{Z}$. We denote by $\Phi^k(X)$ the abelian group of all k-functions on X. For completeness, we define $\Phi^k(X) = \{0\}$ for $k < 0$.

Remark 1.1.2. *A more meticulous author would use here notation $\Phi^k(X;\mathbb{Z})$. For our purposes the coefficients play very little role before discussion on orientation. Thus we fix $\mathbb{Z}$ as our coefficient ring; see Spanier [Spa66] and* Borel-book *for more general treatment. Note, however, that replacing the coefficients ring $\mathbb{Z}$ by the field $\mathbb{R}$, we obtain vector spaces $\Phi^k(X;\mathbb{R})$.*

Remark 1.1.3. *In some sources (e.g. Massey [Mas78]), k-functions are assumed to have finitely many values. This restriction plays, however, no role in our arguments. As a particular consequence of the restriction to finitely many values is that a finitely valued k-function $\phi\colon X^{k+1} \to \mathbb{Z}$ has a unique representation*

$$\phi = \lambda_1 \chi_{A_1} + \cdots + \lambda_m \chi_{A_m}$$

with $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and the pair-wise disjoint sets $A_1, \ldots, A_m \subset X^{k+1}$.

1.1.1 Local equivalence of k-functions

It is crucial to notice that there is no assumption on continuity of k-functions or for the map $f\colon X \to Y$. The topology of X comes forth in the notion of local triviality, which is a key concept in the theory.

For the definition, we denote

$$\Delta^k_X = \{(x, \ldots, x) \in X^{k+1} \colon x \in X\}$$

the *k-diagonal of X*; note that, trivially, $\Delta^k_X \subset X^{k+1}$ for each $k \in \mathbb{N}$, $\Delta^0_X = X$, and $\Delta^k_X = \emptyset$ for $k < 0$.

Definition 1.1.4. Two k-functions $\phi\colon X^{k+1} \to \mathbb{Z}$ and $\psi\colon X^{k+1} \to \mathbb{Z}$ are *locally equivalent* if there exists a neighborhood W of Δ^k_X in X^{k+1} for which

$$\phi|_W = \psi|_W.$$

A k-function ϕ is *locally trivial* if ϕ is equivalent to the zero function $X^{k+1} \to \mathbb{Z}$, $(x_1, \ldots, x_{k+1}) \mapsto 0$.

The following lemma gives a partial answer why this terminology is used.

Lemma 1.1.5. *Let ϕ and ψ be k-functions on X. Then ϕ and ψ are locally equivalent if and only if for each $x \in X$ there exists a neighborhood U of x in X for which $\phi|_{U^{k+1}} = \psi|_{U^{k+1}}$.*

Proof. Suppose ϕ and ψ are locally equivalent and $x \in X$. By local equivalence, there exists a neighborhood W of Δ^k_X for which $\phi|_W = \psi|_W$. Since the product topology of X^{k+1} is generated by products of open sets in X, there exists a neighborhood U of x for which $U^{k+1} \subset W$. Then $\psi|_{U^{k+1}} = \phi|_{U^{k+1}}$.

To the other direction, let, for each $x \in X$, the set U_x be a neighborhood of x in X for which $\phi|_{U_x^{k+1}} = \psi|_{U_x^{k+1}}$, and set $W = \bigcup_{x\in X} U_x^{k+1}$. Then W is a neighborhood of Δ_X^k in X^{k+1} and $\phi|_W = \psi|_W$. Thus ϕ and ψ are locally equivalent. □

It is vital to not confuse locally trivial functions with the zero functions.

Example 1.1.6. *Let $X = \{x, y\}$ be a Hausdorff space consisting of two points. Then the function $\phi\colon X^{1+1} \to \mathbb{Z}$ defined by $\phi(x, y) = \phi(y, x) = 1$, $\phi(x, x) = \psi(y, y) = 0$ is a locally trivial 1-function, since the 1-diagonal Δ_X^1 is open in X^2.*

In what follows, we denote

$$\Phi_0^k(X) = \{\phi \in \Phi^k(X)\colon \phi \text{ is locally trivial}\}.$$

Observation 1.1.7. [1] *Local equivalence of k-functions is an equivalence relation.*

Observation 1.1.8. *Two k-functions ϕ and ψ in $\Phi^k(X)$ are locally equivalent if and only if $\phi - \psi$ is locally trivial. Furthermore, $\Phi_0^k(X)$ is a (necessarily normal) subgroup of $\Phi^k(X)$.*

1.1.2 Support of a k-function

The chains in compactly supported Alexander–Spanier cohomology are equivalence classes of compactly supported k-functions. For this reason, we introduce now the notion of a support $\mathrm{spt}(\phi)$ of a k-function ϕ.

Definition 1.1.9. Let $k \in \mathbb{N}$. A k-function $\phi\colon X^k \to \mathbb{Z}$ is *not supported at* $x \in X$ if there exists a neighborhood $U \subset X$ of x for which $\phi|_{U^k} = 0$. The set $\mathrm{null}(\phi) = \{x \in X : \phi \text{ is not supported at } x\}$ the *nullset of* ϕ. The complement of $\mathrm{null}(\phi)$ in X is the *support* $\mathrm{spt}\,\phi$ *of* ϕ *in* X.

Now it is important to notice that the nullset and the support of a k-function are subsets of the underlying space X and not of the product space X^{k+1}. Note also that Example 1.1.6 gives an easy example of a non-zero k-functions having empty support.

Observation 1.1.10. *Let $\phi \in \Phi^k(X)$. Then the nullset* $\mathrm{null}(\phi)$ *is open and the support* $\mathrm{spt}(\phi)$ *is closed. If $k = 0$, the support* $\mathrm{spt}(\phi)$ *is the usual support of a function, that is,* $\mathrm{spt}(\phi) = \overline{\{x \in X : \phi(x) \neq 0\}}$.

[1]The observations are worthy of their name, easy to prove from definitions. Facts, on the other hand, may need elaborate arguments.

Locally equivalent k-functions have the same support. We state this as a lemma.

Lemma 1.1.11. *Let $\phi, \psi \in \Phi^k(X)$ be locally equivalent k-functions. Then* $\mathrm{spt}(\phi) = \mathrm{spt}(\psi)$.

Proof. The claim is equivalent to the claim $\mathrm{null}(\phi) = \mathrm{null}(\psi)$. Since local equivalence is an equivalence relation, it suffices to show that $\mathrm{null}(\phi) \subset \mathrm{null}(\psi)$.

Let $x \in \mathrm{null}(\phi)$. Then there exists a neighborhood U of x for which $\phi|_{U^{k+1}} = 0$. Since ϕ and ψ are locally equivalent there exists a neighborhood W of Δ_X^k for which $\phi|_W = \psi|_W$. By the definition of product topology, there exists a neighborhood V of x for which $V^{k+1} \subset W \cap U^{k+1}$. Thus $\psi|_{V^{k+1}} = 0$ and $x \in \mathrm{null}(\psi)$. □

Definition 1.1.12. A k-function $\phi\colon X^{k+1} \to \mathbb{Z}$ is *compactly supported* if $\mathrm{spt}(\phi)$ is compact.

We denote

$$\Phi_c^k(X) = \{\phi \in \Phi^k(X) \colon \mathrm{spt}(\phi) \text{ is compact}\}.$$

Observation 1.1.13. *For each $k \in \mathbb{Z}$, $\Phi_c^k(X)$ is a subgroup of $\Phi^k(X)$.*

1.1.3 Coboundary

The chains in Alexander–Spanier theory are given by the equivalence classes in $\Phi^k(X)/\Phi_0^k(X)$. As a preparatory step we consider a coboundary operator on the level of k-functions.

Definition 1.1.14. The *coboundary operator for k-functions* is the homomorphism $\delta^k\colon \Phi^k(X) \to \Phi^{k+1}(X)$ defined by

$$\delta^k(\phi)(x_1, \ldots, x_{k+2}) = \sum_{\ell=1}^{k+2} (-1)^{\ell+1} \phi(x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_{k+2}),$$

where $x_1, \ldots, x_{k+2} \in X$. For completeness, we define $\delta^k = 0\colon \Phi^k(X) \to \Phi^{k+1}(X)$ for $k < 0$.

Convention 1.1.15. *To simplify notation, we denote the homomorphism δ^k simply by δ to unless it is important to emphasize the domain and range.*

Example 1.1.16. *Let X be a point, that is, $X = \{a\}$. We calculate $\delta\colon \Phi^k(X) \to \Phi^{k+1}(X)$ in this case.*

For each $k \in \mathbb{N}$, $X^k = \{(a, \ldots, a)\}$ is also a point and hence $\Phi^k(X)$ is isomorphic to $\mathbb{Z}$, where the isomorphism is $\mathbb{Z} \mapsto \Phi^k(X)$, $m \mapsto ((a, \ldots, a) \mapsto m)$. Let $\phi_k \in \Phi^k(X)$ be the generator of $\Phi^k(X)$ satisfying $\phi_k(a, \ldots, a) = 1$.

For each $k \in \mathbb{N}$, we have

$$\begin{aligned}
\delta\phi_k(x_1,\ldots,x_{k+2}) &= \sum_{\ell=1}^{k+2}(-1)^{\ell+1}\phi_k(x_1,\ldots,\widehat{x}_\ell,x_{k+2}) \\
&= \left(\sum_{\ell=1}^{k+2}(-1)^{\ell+1}\right)\phi_k(a,\ldots,a) = \frac{1+(-1)^{k+1}}{2} \\
&= \frac{1+(-1)^{k+1}}{2}\phi_{k+1}(x_1,\ldots,x_{k+2}).
\end{aligned}$$

Thus

$$\delta\phi_k = \begin{cases} \phi_{k+1}, & k \text{ is odd,} \\ 0, & k \text{ is even.} \end{cases}$$

In particular, δ is an isomorphism for k odd and the zero map for k even.

Example 1.1.17. *Let X be a space and $F \in \Phi^0(X)$, that is, $F\colon X \to \mathbb{Z}$ is a function. Then*

$$\delta F(x,y) = F(y) - F(x)$$

for each $x, y \in X$. Similarly,

$$\begin{aligned}
\delta^2 F(x,y,z) &= \delta F(y,z) - \delta F(x,z) + \delta F(x,y) \\
&= F(z) - F(y) - (F(z) - F(x)) + (F(y) - F(x)) = 0
\end{aligned}$$

for all $x, y, z \in X$. Note that, condition $\delta F(x,y) = 0$ implies

$$F(y) = F(x)$$

for each $x, y \in X$, that is, F is a constant function.

Remark 1.1.18. *The previous example is one of the reasons why we do not take the homology of the complex $\Phi_c^!(X)$ as the compactly supported Alexander–Spanier cohomology of X. Indeed, the homology of the complex $\Phi_c^!(X) = \left(\Phi_c^k(X), \delta\right)_{k\in\mathbb{Z}}$ does not satisfy the additivity axiom, which states that the homology of a disjoint union is a direct sum.*

Example 1.1.19. *For $\psi \in \Phi^1(X)$ condition $\delta\psi = 0$ gives the equation*

$$0 = \delta\psi(x,y,z) = \psi(y,z) - \psi(x,z) + \psi(x,y)$$

i.e. the cocycle condition

$$\psi(x,z) = \psi(x,y) + \psi(y,z)$$

for each $x, y, z \in X$.

A typical calculation, common to all homology/cohomology theories, shows that $\delta\delta = 0$. We leave the verification of this fact to the interested reader.

Observation 1.1.20. *For each $k \in \mathbb{Z}$, $\delta^{k+1} \circ \delta^k = 0$, that is,*

$$\delta^{k+1}(\delta^k \phi)(x_1, \ldots, x_{k+2}) = 0$$

for every $x_1, \ldots, x_{k+2} \in X$.

A fundamental observation is that the coboundary of a k-function has smaller support. We formalize this as follows.

Lemma 1.1.21. *Let $k \in \mathbb{Z}$ and $\phi \in \Phi^k(X)$. Then $\mathrm{spt}(\delta\phi) \subset \mathrm{spt}(\phi)$.*

Proof. Let $x \in \mathrm{null}(\phi)$. Then there exists a neighborhood $U \subset X$ of x for which $\phi|_{U^{k+1}} = 0$. Thus, for $x_1, \ldots, x_{k+2} \in U$, we have

$$\begin{aligned} \delta^k \phi(x_1, \ldots, x_{k+2}) &= \sum_{\ell=1}^{k+2} (-1)^{\ell+1} \phi(x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_{k+2}) \\ &= \sum_{\ell=1}^{k+2} (-1)^{\ell+1} \phi|_{U^{k+1}}(x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_{k+2}) = 0. \end{aligned}$$

Thus $x \in \mathrm{null}(\delta^k(\phi))$. □

Corollary 1.1.22. *For each $k \in \mathbb{Z}$,*

$$\delta\Phi_0^k(X) \subset \Phi_0^k(X) \quad \textit{and} \quad \delta\Phi_c^k(X) \subset \Phi_c^k(X).$$

Example 1.1.23. *Let $X = \mathbb{R}$ and $F \colon \mathbb{R} \to \mathbb{Z}$,*

$$x \mapsto \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

In particular, $\mathrm{spt}(F) = [0, \infty)$. On the other hand,

$$\delta F(x,y) = F(y) - F(x) = \begin{cases} 1, & x < 0 \leq y, \\ 0, & x, y \geq 0 \text{ or } x, y < 0, \\ -1, & x \geq 0 \text{ and } y < 0. \end{cases}$$

Suppose $x \neq 0$. Then (x,x) is contained either in $(0,\infty)\times(0,\infty)$ or $(-\infty,\infty)$ and $\delta F|_{U^2} = 0$ where U is either of these open sets. Thus $\mathrm{null}(\delta F) \supset \mathbb{R}\setminus\{0\}$. Since clearly $\{0\} \subset \mathrm{spt}(\delta F)$, we conclude that $\mathrm{spt}(\delta F) = \{0\}$.

Having the coboundary operator and notion of support at our disposal, we have a topological characterization for the kernel of δ^0. Recall that a function $F \colon X \to \mathbb{Z}$ is *locally constant* if for each $x \in X$ there exists a neighborhood U of x in X for which $F|_U \colon U \to \mathbb{Z}$ is constant.

Lemma 1.1.24. *Let $F \in \Phi^0(X)$, that is, a function $F \colon X \to \mathbb{Z}$. Then $\mathrm{spt}(\delta F) = \emptyset$ if and only if $F \colon X \to \mathbb{Z}$ is locally constant.*

Proof. Suppose first that $\mathrm{spt}(\delta^0 F) = \emptyset$, that is, $(\delta F) = X$. Let $x \in X$. Then there exists a neighborhood U of x in X for which $\delta F|U^2 = 0$. Thus $F(y) - F(x) = \delta F(x,y) = 0$ for all $y \in U$. Hence $F|_U$ is constant. The other direction is similar. □

1.2 Algebraic intermission

We recall some basic algebraic notions and facts. Let G be an abelian group.

1.2.1 Quotient spaces

Given a subgroup $H \subset G$, the coset $g + H$ of $g \in G$ is the set $\{v + w \in G \colon h \in H\}$. The set G/H of all cosets $\{g + H \colon g \in G\}$ is a partition of G and it induces an equivalence relation $\sim_H$ on G; we define $g \sim_H g'$ if and only if $g - g' \in H$.

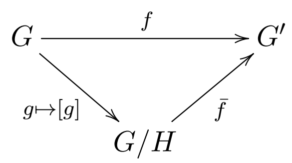
Observation 1.2.1. *The addition* $+\colon G/H \times G/H \to G/H$

$$(g + H) + (g' + H) = (g + g') + H$$

for $g + H, g' + H \in G/H$, *is well-defined and* $(G/H, +)$ *is an abelian group.*

Convention 1.2.2. *Typically the element* $g + H$ *of* G/H *is denoted also by* $[g]$ *suppressing the subgroup* H *from the notation. We follow this convention in forthcoming sections.*

Observation 1.2.3. *Let* $f\colon G \to G'$ *be a homomorphism and* $H < \ker f$ *a subgroup. Then there exists a unique homomorphism* $\bar{f}\colon G/H \to G'$ *satisfying*



Moreover $\bar{f}$ *is an isomorphism if* $fG = G'$ *and* $H = \ker f$. *(Note that no condition on normality is needed, since* G *is abelian.)*

1.2.2 Chain complexes

Definition 1.2.4. A sequence $G_\# = (G_k, \alpha_k)_{k\in\mathbb{Z}}$ of abelian groups and homomorphisms $\alpha_k \colon G_k \to G_{k+1}$ is a *chain complex* if $\alpha_{k+1} \circ \alpha_k = 0$. The homology $H_*(G_\#)$ of $G_\#$ is the sequence $(H_k(G_\#))_{k\in\mathbb{Z}}$ where

$$H_k(G_\#) = \ker \alpha_k \Big/ \operatorname{im} \alpha_{k-1}$$

for each $k \in \mathbb{Z}$.

Example 1.2.5. *The sequences*

$$\Phi^!(X) = (\Phi^k(X), \delta^k)_{k\in\mathbb{Z}} \quad \textit{and} \quad \Phi^!_c(X) = (\Phi^k_c(X), \delta^k)_{k\in\mathbb{Z}}$$

are chain complexes.

Remark 1.2.6. *Note that* $\operatorname{im} \alpha_{k-1} \subset \ker \alpha_k$*, since* $\alpha_k \circ \alpha_{k+1} = 0$.

Definition 1.2.7. Let $G_\# = (G_k, \alpha_k)$ and $G'_\# = (H_k, \beta_k)$ be chain complexes. A sequence $f_\# = (f_k \colon G_k \to G'_k)_{k \in \mathbb{Z}}$ is a *chain map* $f_\# \colon G_\# \to G'_\#$ if

$$\begin{array}{ccc} G_k & \xrightarrow{\alpha_k} & G_{k+1} \\ {\scriptstyle f_k}\big\downarrow & & \big\downarrow{\scriptstyle f_{k+1}} \\ G'_k & \xrightarrow[\beta_k]{} & G'_{k+1} \end{array}$$

commutes.

Example 1.2.8. *Pull-back homomorphisms* $f^! \colon \Phi^k(Y) \to \Phi^k(X)$*, and their restrictions* $f^! \colon \Phi^k_0(Y) \to \Phi^k_0(X)$ *and* $f^! \colon \Phi^k_c(Y) \to \Phi^k_c(X)$*, are chain maps.*

Lemma 1.2.9. *Let* $G_\#$ *and* $G'_\#$ *be chain complexes and* $f_\# \colon G_\# \to G'_\#$ *a chain map. Then, for each* $k \in \mathbb{Z}$*, there exists a well-defined linear map* $f_* := f_{*k} \colon H_k(G_\#) \to H_k(G'_\#)$ *satisfying* $f_{*k}([v]) = [f_k(v)]$.

Proof. Since f_* is a chain map, $f_k(\ker \alpha_k) \subset \ker \beta_k$ and $\operatorname{im} f_k \circ \alpha_{k-1} \subset \operatorname{im} \beta_{k-1}$. Thus we have a diagram

$$\begin{array}{ccc} \ker \alpha_k & \xrightarrow{f_k} & \ker \beta_k \\ {\scriptstyle v \mapsto [v]}\big\downarrow & \searrow & \big\downarrow{\scriptstyle w \mapsto [w]} \\ \ker \alpha_k / \operatorname{im} \alpha_{k-1} & \overset{}{\underset{f_{*k}}{\dashrightarrow}} & \ker \beta_k / \operatorname{im} \beta_{k-1} \end{array}$$

□

1.2.3 Exact sequences

Definition 1.2.10. Let A, B, and C be abelian groups and $f \colon A \to B$ and $g \colon B \to C$ homomorphisms. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* B if $\ker g = \operatorname{im} f$.

A basic result on exact sequences is the *Five Lemma*[2].

Fact 1.2.11. *Let*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \big\downarrow{\scriptstyle h_1} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle h_2} & & \big\downarrow{\scriptstyle h_3} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle h_4} & & \big\downarrow{\scriptstyle h_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

[2]Typically the introduction of the Five Lemma is followed by saying "Chace the diagram."

be a commutative diagram of abelian groups and homomorphisms having exact rows. Suppose that h_2 and h_4 are isomorphisms. Suppose also that h_1 is surjective and h_5 is injective. Then h_3 is an isomorphism.

A sequence

$$A_\# \xrightarrow{f_\#} B_\# \xrightarrow{g_\#} C_\#$$

of chain complexes and chain maps is *exact* if

$$A_k \xrightarrow{f_k} B_k \xrightarrow{g_k} C_k$$

is exact.

Definition 1.2.12. A sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of abelian groups and homomorphisms is a *short exact sequence* if it is exact at A, B, and C.

A sequence

$$\cdots \longrightarrow A_{k-1} \longrightarrow A_k \longrightarrow A_{k+1} \longrightarrow \cdots$$

of abelian groups and homomorphisms is a *long exact sequence* if the sequence is exact at each A_k.

Example 1.2.13. *Let $f\colon V \to W$ be a linear map. The sequence*

$$0 \longrightarrow \ker f \hookrightarrow V \xrightarrow{f} \operatorname{im} f \longrightarrow 0$$

is a short exact sequence.

The short and long exact sequences of chain complexes are defined similarly.

Observation 1.2.14. *Let $G_\# = (G_k, \alpha_k)$ be a chain complex. Then $H_k(G_\#) = 0$ if and only if the sequence*

$$G_{k-1} \xrightarrow{\alpha_{k-1}} G_k \xrightarrow{\alpha_k} G_{k+1}$$

is exact at V_k

Observation 1.2.15. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

be an exact sequence. Then f is an isomorphism.

Indeed, since the sequence is exact, $\ker f = \{0\}$ and $\operatorname{im} f = B$.

A beautiful fact, which motivates for us the whole discussion in this section, is that short exact sequence of chain complexes yields a long exact sequence in (co)homology.

Fact 1.2.16. *Let*

$$0 \to A_\# \xrightarrow{f_\#} B_\# \xrightarrow{g_\#} C_\# \longrightarrow 0$$

be a short exact sequence of chain complexes (and chain maps). Then there exists homomorphisms $\partial_k \colon H_k(C_\#) \to H_{k+1}(A_\#)$ (so-called connecting homomorphisms*) for which*

$$\cdots \longrightarrow H_k(A_\#) \xrightarrow{f_*} H_k(B_\#) \xrightarrow{g_*} H_k(C_\#) \xrightarrow{\partial_k} H_{k+1}(A_\#) \longrightarrow \cdots$$

is a long exact sequence.

Idea of the proof: Show that the homomorphism $H_k(C_\#) \to H_{k+1}(A_\#)$, $[c] \mapsto [f_\#^{-1}\beta_k g_\#^{-1}c]$ is well-defined and satisfies the required properties by chasing the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_k & \longrightarrow & B_k & \xrightarrow{g_\#} & C_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta_k} & & \downarrow & & \\
0 & \longrightarrow & A_{k+1} & \xrightarrow{f_\#} & B_{k+1} & \longrightarrow & C_{k+1} & \longrightarrow & 0
\end{array}$$

□

Finally, we remark the construction of the long exact sequence is natural.

Fact 1.2.17. *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_\# & \xrightarrow{f_\#} & B_\# & \xrightarrow{g_\#} & C_\# & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle h^1_\#} & & \uparrow{\scriptstyle h^2_\#} & & \uparrow{\scriptstyle h^3_\#} & & \\
0 & \longrightarrow & A'_\# & \xrightarrow{f'_\#} & B'_\# & \xrightarrow{g'_\#} & C'_\# & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain complexes and chain maps having exact rows. Then the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_k(A_\#) & \xrightarrow{f_*} & H_k(B_\#) & \xrightarrow{g_*} & H_k(C_\#) & \xrightarrow{\partial_k} & H_{k+1}(A_\#) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle h^1_*} & & \uparrow{\scriptstyle h^2_*} & & \uparrow{\scriptstyle h^3_*} & & \uparrow{\scriptstyle h^1_*} & & \\
\cdots & \longrightarrow & H_k(A'_\#) & \xrightarrow{f'_*} & H_k(B'_\#) & \xrightarrow{g'_*} & H_k(C'_\#) & \xrightarrow{\partial_k} & H_{k+1}(A'_\#) & \longrightarrow & \cdots
\end{array}$$

commutes.

1.2.4 Products and sums

Given a family of sets $(X_i)_{i\in I}$, formally the elements of the product $\prod_{i\in I} X_i$ are functions $f\colon I \to \bigcup_{i\in I} X_i$ satisfying $f(i) \in X_i$ for each $i \in I$. In what follows, however, we denote the elements of $\prod_{i\in I} X_i$ as ordered families $(x_i)_{i\in I} \in \prod_{i\in I} X_i$, where $x_i \in X_i$ for each $i \in I$.

Given abelian groups $(G_i)_{i\in I}$, the *direct product of the groups* $(G_i)_{i\in I}$ is the abelian group $\prod_{i\in I} G_i$ with group operation given by

$$(g_i)_{i\in I} + (g_i')_{i\in I} := (g_i + g_i')_{i\in I}.$$

for all $(g_i)_{i\in I}, (g_i')_{i\in I} \in \prod_{i\in I} G_i$.

The *direct sum* $\bigoplus_{i\in I} G_i$ *of abelian groups* $\{G_i\}_{i\in I}$ is the subgroup of $\prod_{i\in I} G_i$ consisting of the elements $(g_i)_{i\in I}$ having finite support, that is, elements $(g_i)_{i\in I}$ for which the set $\{i \in I : g_i \neq e_{G_i}\}$ is finite.

We finish with an observation on direct sums of chain complexes.

Observation 1.2.18. *The complex* $\left(C_c^k(U) \oplus C_c^k(V), d \oplus d\right)_{k\in\mathbb{Z}}$ *is a well-defined chain complex and the homomorphism*

$$H_k(C_c^*(U) \oplus C_c^*(V)) \to H_c^k(U) \oplus H_c^k(V), [(a,b)] \mapsto ([a],[b]),$$

is a well-defined isomorphism.

Remark 1.2.19. *All the results in Section 1.2 hold if we consider, instead of abelian groups and group homomorphsms, R-modules and R-module homomorphisms, where R is a commutative ring; note that abelian groups are* $\mathbb{Z}$*-modules. We do not need this generality in what follows.*

1.3 Cochains and cohomology

The Alexander–Spanier k-cochains are defined as equivalence classes of k-functions modulo locally trivial k-functions. The formal definition reads as follows.

Definition 1.3.1. The elements of the quotient space

$$C^k(X) = \Phi^k(X)/\Phi_0^k(X)$$

are called *(Alexander–Spanier) k-cochains* and the space $C^k(X)$ as the *space of (Alexander–Spanier) k-cochains in X*. Similarly, the elements of

$$C_c^k(X) = \Phi_c^k(X)/\Phi_0^k(X)$$

are *compactly supported k-cochains* and $C_c^k(X)$ is the *space of compactly supported (Alexander–Spanier) k-cochains in X*.

Remark 1.3.2. *Since $\Phi^0_0(X) = \{0\}$ by Observation 1.1.10, the quotient maps $\Phi^0(X) \to C^0(X)$ and $\Phi^0_c(X) \to C^0_c(X)$, $\phi \mapsto [\phi]$, are isomorphisms.*

Remark 1.3.3. *Let ϕ and ψ be k-functions so that $[\phi] = [\psi] \in C^k(X)$. Then $\phi - \psi \in \Phi^k_0(X)$. Thus, by Lemma 1.1.11, $\mathrm{spt}(\phi) = \mathrm{spt}(\psi)$.*

Definition 1.3.4. The support $\mathrm{spt}(c)$ of a cochain $c \in C^k(X)$ is $\mathrm{spt}(\phi)$ for a k-function (and hence any k-function) ϕ in c.

1.3.1 Coboundary

It is a direct consequence of Lemma 1.1.21 that the coboundary operator $\delta^k \colon \Phi^k(X) \to \Phi^{k+1}(X)$ descends to a coboundary operator $d^k \colon C^k(X) \to C^{k+1}(X)$ on cochains. We leave the details of this fact to the interested reader.

Lemma 1.3.5. *For each $k \in \mathbb{Z}$ there exists a linear map $d^k \colon C^k(X) \to C^{k+1}(X)$ for which $d^k[\phi] = [\delta^k(\phi)]$ for all $\Phi^k(X)$. In particular, $d^{k+1} \circ d^k = 0$ for each k.*

Proof. Since $\delta\Phi^k_0(X) \subset \Phi^{k+1}_0(X)$, there exists a unique homomorphism $d^k \colon \Phi^k(X)/\Phi^k_0(X) \to \Phi^{k+1}(X)/\Phi^{k+1}_0(X)$ for which the digram

$$\begin{array}{ccc} \Phi^k(X) & \xrightarrow{\quad\delta^k\quad} & \Phi^{k+1}(X) \\ {\scriptstyle \phi\mapsto[\phi]}\downarrow & \searrow & \downarrow{\scriptstyle \phi\mapsto[\phi]} \\ \Phi^k(X)/\Phi^k_0(X) & \xrightarrow{\quad d^k\quad} & \Phi^{k+1}(X)/\Phi^{k+1}_0(X) \end{array}$$

commutes. Moreover, for each $\phi \in \Phi^k(X)$,

$$(d^{k+1} \circ d^k)([\phi]) = d^{k+1}([\delta^k\phi]) = [\delta^{k+1}\delta^k\phi] = 0.$$

Thus $d^{k+1} \circ d^k = 0$. □

Combining Lemma 1.3.5 and Corollary 1.1.22 we obtain the following important observation.

Lemma 1.3.6. *For each $k \in \mathbb{Z}$, the restriction*

$$d^k_c = d^k|_{C^k_c(X)} \colon C^k_c(X) \to C^{k+1}_c(X).$$

of d^k is well-defined and the sequence

$$C^\#_c(X) = (C^k_c(X), d^k_c)_{k\in\mathbb{Z}}$$

is a chain complex.

Proof. For the first claim, it suffices to observe that, for each $\phi \in \Phi_c^k(X)$,

$$\mathrm{spt}(d[\phi]) = \mathrm{spt}([\delta\phi]) = \mathrm{spt}(\delta\phi) \subset \mathrm{spt}(\phi)$$

is compact. The second claim follows from the fact that $d^{k+1}d^k = 0$. $\square$

Convention 1.3.7. *Although it would formally more appropriate to denote the restriction $d^k|_{C_c^k(X)}$ with a different symbol, in what follows, we merely denote $d\colon C_c^k(X) \to C_c^{k+1}(X)$ and $C_c^\#(X) = (C_c^k(X), d)_{k\in\mathbb{Z}}$.*

Definition 1.3.8. A cochain $c \in C_c^k(X)$ is a *k-cocycle* if $dc = 0$, and a *k-coboundary* if there exists $b \in C_c^{k-1}(X)$ for which $db = c$.

We record a simple observation on 0-cocycles, for further use, as lemma.

Lemma 1.3.9. *Let $F\colon X \to \mathbb{Z}$ be a 0-function for which $d[F] = 0$. Then F is locally constant.*

Proof. Let $x \in X$. Since $[\delta F] = d[F] = 0$, there exists a neighborhood U of x for which $\delta F|_{U^2} = 0$. Then, for each $y \in U$,

$$F(y) - F(x) = \delta F(x, y) = 0.$$

Thus F is locally constant. $\square$

Example 1.3.10. *Let $[\psi] \in C_C^1(X)$ be a cocycle, i.e. $[\delta\psi] = d[\psi] = 0$. Let $x \in X$. Since $\delta\psi$ is locally trivial, there exists a neighborhood U of x for which $\delta\phi|_{U^3} = 0$. Thus, for $x, y, z \in U$,*

$$\phi(x, y) = \phi(x, z) + \phi(z, y);$$

cf. Example 1.1.19.

1.3.2 Cohomology $H_c^*(\cdot)$

Since $\ker d_c^k \subset \mathrm{im}\, d_c^{k-1}$ for each $k \in \mathbb{Z}$, we have well-defined quotient spaces

$$H_c^k(X) = \ker d_c^k \big/ \mathrm{im}\, d_c^{k-1} = \frac{\{[\phi] \in C_c^k(X)\colon d^k[\phi] = 0\}}{\{d^{k-1}[\psi] \in C_c^k(X)\colon [\psi] \in C_c^{k-1}(X)\}}$$

for each $k \in \mathbb{Z}$.

Heuristically, $H_c^k(X)$ measures the amount of non-trivial solutions of the equation $d^k[\phi] = d^k[\phi']$ i.e. the number different solutions $[\phi]$ and $[\phi']$ for the equation $d[\phi] = 0$ for which the equation $[\phi] - [\phi'] = d^{k-1}[\psi]$ does not hold. In terminology of Section 1.2, $H_c^k(X)$ is the homology $H_k(C_c^\#(X))$ of the chain complex $C_c^\#(X)$.

Definition 1.3.11. For $k \in \mathbb{Z}$, the abelian group $H_c^k(X)$ is the *kth compactly supported Alexander–Spanier cohomology group of X*. The elements of $H_c^k(X)$ are called *compactly supported Alexander–Spanier cohomology classes of X*.

Examples

We consider now some standard examples. Starting from one point, as is commonly done.

Example 1.3.12. *Let X be a point. Then*

$$H_c^k(X) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, we observe first that in this case, $\Phi_0^k(X) = \{0\}$ and $\Phi^k(X) = \Phi_c^k(X)$. Thus we may identify $C_c^k(X)$ with $\Phi_c^k(X)$ and $d^k \colon C_c^k(X) \to C_c^{k+1}(X)$ with $\delta^k \colon \Phi_c^k(X) \to \Phi_c^k(X)$. By Example 1.1.16, the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^0(X) & \xrightarrow{d} & C^1(X) & \xrightarrow{d} & C^2(X) & \xrightarrow{d} & C^3(X) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \cdots \end{array}$$

commutes. Thus, $\ker d^0 \cong \mathbb{Z}$ and $\operatorname{im} d^k = \ker d^{k+1}$ for $k \neq 0$.

Example 1.3.13. *Let X be a compact connected space. Then $H_c^0(X) \cong \mathbb{Z}$.*

Indeed, let $F \colon X \to \mathbb{Z}$ be the constant function 1. Since X is compact, $F \in \Phi_c^0(X)$. Since $\operatorname{spt}(F) \neq \emptyset$, $[F] \neq 0$ in $\Phi_c^0(X)/\Phi_0^0(X) = C_c^0(X)$. Since $d[F] = [\delta F] = 0$, $[F] \in \ker d^0$. Finally, since $C_c^{-1}(X) = \{0\}$, we have that $\operatorname{im} d^{-1} = 0$ and $[\phi] \neq 0$ in $H_c^0(X)$. Thus $H_c^0(X) \neq \{0\}$.

We show now that if $c \in H_c^0(X)$ then $c = m[F]$ for some $m \in \mathbb{Z}$. The claim follows from this observation. Let $[c] \in H_c^0(X)$ and $G \in \Phi_c^0(X)$ for which $c = [G]$ in $H_c^0(X)$. Since $[\delta G] = d[G] = dc = 0$ and $G \colon X \to \mathbb{Z}$ is a function, we conclude that G is localy constant function. Since X is connected, G is a constant function and hence an integer multiple of F.

Example 1.3.14. *Let X be a connected non-compact space. Then $H_c^0(X) = 0$.*

Indeed, let $[c] \in H_c^0(X)$ and $F \in \Phi_c^0(X)$ a representative of c. Since $[\delta F] = dc = 0$, we have $\delta F \in \Phi_0^1(X)$. Thus F is a locally constant function. Since X is connected, F is a constant function. Since $\operatorname{spt}(F)$ is compact and X is non-compact, we conclude that $F = 0$. Thus $H_c^0(X) = 0$.

1.3.3 $H_c^1(\mathbb{R}) \cong \mathbb{Z}$

By Example 1.3.14, $H_c^0(\mathbb{R}) = 0$. We show next that $H_c^1(\mathbb{R}) \cong \mathbb{Z}$; note that it is not clear at this point that $H_c^k(\mathbb{R}) = 0$ for $k > 1$ (although that will be the case). For the importance of this result (and the length of its proof), we record this example as a theorem.

Theorem 1.3.15. *We have*

$$H^1_c(\mathbb{R}) \cong \mathbb{Z}.$$

To simplify the proof, we separate an auxiliary lemma.

Lemma 1.3.16. *Let* $[\psi] \in C^1(\mathbb{R})$ *be a cocycle, that is,* $d[\psi] = 0$*. Then there exists a function* $F_\psi \in \Phi^0(\mathbb{R})$ *for which* $[\psi] = d[F_\psi]$ *in* $C^1(\mathbb{R})$*.*

Proof. Let $\psi \in \Phi^1(\mathbb{R})$ be a representative of c. Since $\delta\psi$ is locally trivial, there exists a covering $\mathscr{U}$ of $\mathbb{R}$ by open intervals so that $(\delta\psi)|_{U^3} = 0$ for each $U \in \mathscr{U}$. Then, for each $U \in \mathscr{U}$ and $x, y, z \in U$,

$$\psi(x, y) = \psi(x, z) + \psi(z, y). \tag{1.3.1}$$

We define a function $F_\psi \colon \mathbb{R} \to \mathbb{R}$ as follows. We fix first a basepoint $y_0 \in \mathbb{R}$. For $y \in \mathbb{R}$, let $y_1, \ldots, y_k = y$ be a (monotone) sequence having the property that, for each $i = 1, \ldots, k$, there exists $U_i \in \mathcal{U}$ so that $[y_{i-1}, y_i] \subset U_i$. We set

$$F_\psi(y) = \sum_{i=1}^{k} \psi(y_{i-1}, y_i).$$

By (1.3.1), the value $F_\psi(y)$ does not depend on the choice of the sequence $y_1, \ldots, y_k$.

It remains to show that ψ and δF_ψ are locally equivalent. Let $U \in \mathscr{U}$ and $x, y \in U$. We may assume that $y_0 < y < x$, the other cases are similar. Let $y_1, \ldots, y_k = x$ be a monotone sequence defining $F_\psi(x)$ as above. Then $y_1, \ldots, y_k, y$ is a valid sequence to define $F_\psi(y)$. Thus

$$\delta F_\psi(x, y) = F_\psi(y) - F_\psi(x) = \psi(x, y).$$

This completes the proof. □

Proof of Theorem 1.3.15. Let $F \colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of $\mathbb{R}_+$, i.e. the function $F = \chi_{[0,\infty)}$. Since $\mathrm{spt}(\delta F) = \{0\}$, the cochain $c = [\delta F]$ is compactly supported, that is, $c \in C^1_c(\mathbb{R})$. Since $dc = d[\delta F] = [\delta\delta F] = 0$, the cochain c is a cocycle, and c represents a cohomology class in $H^1_c(\mathbb{R})$. It suffices to show that $[c]$ generates $H^1_c(\mathbb{R})$.

Step 1: The class $[c]$ *is non-trivial.* Suppose towards contradiction that $[c] = 0$. Then there exists a cochain $[G] \in C^0_c(\mathbb{R})$ for which $c = d[G]$. Since $c = [\delta F]$, we conclude that $d[F] = d[G]$ in $C^1(X)$. Thus $F - G$ is a constant function by Lemma 1.3.9. Since G is compactly supported, this is contradiction.

Step 2: The class $[c]$ *generates the cohomology group* $H^1_c(\mathbb{R})$*.* Let $[c'] \in H^1_c(\mathbb{R})$ and $\psi \in \Phi^1_c(\mathbb{R})$ a representative of c'. By Lemma 1.3.16, there exists a function $F_\psi \colon \mathbb{R} \to \mathbb{Z}$ for which $[\psi] = d[F_\psi]$. Since ψ and δF_ψ are locally equivalent, we conclude that δF_ψ has compact support and that F_ψ is locally constant in the complement of the support of ψ. We fix $M > 0$ for which the interval $[-M, M]$ contains the support of ψ.

By adding a constant to function F_ψ if necessary, we may assume that $\mathrm{spt}(F_\psi) \subset [-M, \infty)$. Let also $\lambda \in \mathbb{Z}$ be the value of F_ψ in $[M, \infty)$.

Let $f\colon \mathbb{R} \to \mathbb{Z}$ be the function $f = F_\psi - \lambda F$. Then $\mathrm{spt}(f) \subset [-M, M]$. Indeed, for $x < -M$, we have $f(x) = F_\psi(x) - \lambda F(x) = 0$ and, for $x > M$, $f(x) = F_\psi(x) - \lambda F(x) = \lambda - \lambda = 0$.

Since

$$\begin{aligned} c' - \lambda c &= [\psi] - \lambda[\phi] = [\delta F_\psi] - \lambda[\delta F] \\ &= [\delta F_\psi - \lambda \delta F] = [\delta(F_\psi - \lambda F)] = [\delta f] = d[f], \end{aligned}$$

we have

$$[c'] = [c' - \lambda c] + \lambda[c] = [d[f]] + \lambda[c] = \lambda[c]$$

in $H^1_c(\mathbb{R})$. This completes the proof. □

1.4 Pull-back

Typically a continuous mapping induces a pull-back in cohomology. In the case of compactly supported cohomology it is natural that the mapping is also proper. Recall that a continuous mapping $f\colon X \to Y$ is *proper* if each compact set $E \subset Y$ has a compact pre-image $f^{-1}E$.

We develop the pull-back homomorphism in three steps. First, for k-functions, than cochains, and finally for cohomology. Since the construction of the pull-back is standard, the necessary steps are listed as observations. Note that, for k-functions, we do not formally need even continuity.

Definition 1.4.1. Let $f\colon X \to Y$ be a map. The *pull-back homomorphism* $f^!\colon \Phi^k(Y) \to \Phi^k(X)$ is the homomorphism $\phi \mapsto f^!(\phi)$, where $f^!(\phi)\colon X^{k+1} \to \mathbb{R}$ is the function

$$f^!(\phi)(x_1, \ldots, x_{k+1}) = \phi(f(x_1), \ldots, f(x_{k+1})).$$

for each $\phi \in \Phi^k(Y)$ and $x_1, \ldots, x_{k+1} \in X$.

Lemma 1.4.2. *Let $f\colon X \to Y$ be a continuous map and $\phi \in \Phi^k(Y)$. Then* $\mathrm{spt}\, f^!(\phi) \subset f^{-1}\,\mathrm{spt}(\phi)$.

Proof. Let $y \in \mathrm{null}(\phi)$ and $x \in f^{-1}(y)$. Since f is continuous, there exists a neighborhood V of x so that fV is contained in a neighborhood U of y so that $\phi|_{U^{k+1}} = 0$. Hence $f^!(\phi)|_{V^{k+1}} = 0$ and $x \in \mathrm{null}(f^!(\phi))$. Thus $f^{-1}(\mathrm{null}(\phi)) \subset \mathrm{null}(f^!(\phi))$. Hence

$$\mathrm{spt}(f^!(\phi)) = X \setminus \mathrm{null}(f^!(\phi)) \subset X \setminus f^{-1}(\mathrm{null}(\phi)) = f^{-1}\,\mathrm{spt}(\phi).$$

This concludes the proof. □

Corollary 1.4.3. *Let $f\colon X \to Y$ be a continuous map, and let $\phi\colon Y^{k+1} \to \mathbb{Z}$ and $\psi\colon Y^{k+1} \to \mathbb{Z}$ be locally equivalent k-functions. Then $f^!\phi$ and $f^!\psi$ are locally equivalent. In particular, $f^!\Phi^k_0(Y) \subset \Phi^k_0(X)$.*

Corollary 1.4.4. *Let $f\colon X \to Y$ be a proper continuous map. Then $f^!\Phi^k_c(Y) \subset \Phi^k_c(X)$.*

Remark 1.4.5. *The inclusion $\mathrm{spt}(f^!\phi) \subset f^!\,\mathrm{spt}(\phi)$ may be strict. Indeed, let $X = \mathbb{R}^2$, $A = \mathbb{R} \times \{0\}$, $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function $\phi = \chi_{\mathbb{R}^2 \setminus A}$, and $\iota\colon A \hookrightarrow X$ the inclusion. Then $\phi \in \Phi^0(\mathbb{R}^2)$ and $\mathrm{spt}(\phi) = \mathbb{R}^2$. On the other hand, $\iota^!\phi = 0$ and $\mathrm{spt}(\iota^!\phi) = \emptyset$.*

The coboundary operator δ and the pull-back $f^!$ clearly commute.

Observation 1.4.6. *Let $f\colon X \to Y$ be a map and $\phi \in \Phi^k(Y)$. Then*

$$(\delta \circ f^!)(\phi) = (f^! \circ \delta)(\phi).$$

In particular, the pull-back $f^!\colon \Phi^k(Y) \to \Phi^k(X)$ decends as a pull-back $f^\#\colon C^k(Y) \to C^k(X)$.

Lemma 1.4.7. *Let $f\colon X \to Y$ be a continuous map. Then there exists a homeomorphism $f^\#\colon C^k(Y) \to C^k(X)$ satisfying $f^\#[\phi] = [f^!\phi]$. Furthermore, if f is proper, the restriction $f^\#\colon C^k_c(Y) \to C^k_c(X)$ is well-defined.*

Proof. By Corollary 1.4.3, $f^!\Phi^k_0(Y) \subset \Phi^k_0(X)$. Thus there exists a homomorphism $f^\#\colon \Phi^k(Y)/\Phi^k_0(Y) \to \Phi^k(X)/\Phi^k_0(X)$ satisfying

$$\begin{array}{ccc} \Phi^k(Y) & \xrightarrow{\quad f^! \quad} & \Phi^k(X) \\ \downarrow & \searrow & \downarrow \\ \Phi^k(Y)/\Phi^k_0(Y) & \underset{f^\#}{\dashrightarrow} & \Phi^k(X)/\Phi^k_0(X) \end{array}$$

where vertical arrows are quotient maps. Similar application of Corollary 1.4.3 gives also the other claim. □

It is straightforward to see that the coboundary operator d commutes with $f^\#$. We record this as an observation.

Observation 1.4.8. *Let $f\colon X \to Y$ be a continuous map. Then*

$$\begin{array}{ccc} C^k(Y) & \xrightarrow{d^k} & C^{k+1}(Y) \\ {\scriptstyle f^\#}\downarrow & & \downarrow{\scriptstyle f^\#} \\ C^k(X) & \underset{d^k}{\longrightarrow} & C^{k+1}(X) \end{array}$$

commutes. In particular, $f^\#\colon C^\#(Y) \to C^\#(X)$ is a chain map. If in addition the map f is proper, $f^\#\colon C^\#_c(Y) \to C^\#_c(X)$ is a chain map.

Finally, a proper continuous map induces a pull-back in cohomology.

Observation 1.4.9. *Let $f\colon X \to Y$ be a proper continuous map. Then there exists a homomorphism $f^*\colon H_c^k(Y) \to H_c^k(X)$ satisfying $f^*[c] = [f^\# c]$.*

The pull-back f^* is natural in the following sense.

Observation 1.4.10. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be proper continuous mappings. Then*

$$(g \circ f)^* = f^* \circ g^*\colon H_c^*(Z) \to H_c^*(X).$$

Furthermore, $\mathrm{id}^* = \mathrm{id}\colon H_c^*(X) \to H_c^*(X)$. *In particular, if $f\colon X \to Y$ is a homeomorphism, the homomorphism $f^*\colon H_c^*(Y) \to H_c^*(X)$ is an isomorphism.*

We finish this section with two simple observations on inclusions and closed sets.

Lemma 1.4.11. *Let X be a space and $A \subset X$ a closed subset. Then the inclusion $\iota\colon A \hookrightarrow X$ is a proper map. In particular, ι induces the pull-back $(\iota_{XA})^*\colon H_c^*(X) \to H_c^*(A)$.*

Proof. Let $E \subset X$ be a compact set. Then $E \cap A$ is closed in E. Hence $\iota^{-1}(E) = E \cap A$ is compact. Thus ι is proper. □

Lemma 1.4.12. *Let X be a compact connected space and $A \subset X$ a connected closed subset. Then the inclusion $\iota\colon A \to X$ induces an isomorphism $\iota^*\colon H_c^0(X) \to H_c^0(A)$.*

Proof. Since X and A are compact and connected, $H_c^0(X) = \langle[\chi_X]\rangle$ and $H_c^0(A) = \langle[\chi_A]\rangle$, where χ_X and χ_A are characteristic functions of X and A, respectively; note that we tacitly identify $C_c^0(\cdot) = \Phi_c^0(\cdot)$. Since $\iota^\#\chi_X = \chi_A$, we have

$$\iota^*[\chi_X] = [\iota^\#\chi_X] = [\chi_A].$$

Thus the claim follows. □

1.5 Push-forward

Regarding our discussion, the push-forward of cohomology classes is a key notion in what follows. Push-forward induced by an inclusion is an operation familar from many compactly supported cohomology theories. Here, heuristically, on the level of k-functions it can be seen as the zero extension. Formally, however, the most straightforward zero extension leads to unwanted increase of support of k-functions. To overcome the unwanted phenomenon, consider auxiliary neighborhoods of supports. On the level of cochains these choices of supports have no role.

To obtain reasonable theory, some assumptions on the space X are needed. In this section we assume that X is locally compact and Hausdorff.

Inclusions

Let X be a space and $A \subset X$ a subset. In what follows, we denote

$$\iota_{XA} \colon A \to X$$

the inclusion map $A \hookrightarrow X$.

Remark 1.5.1. *The reader may wonder the unnatural order in the subscript. This choice stems from the composition rule, namely, for $A \subset B \subset X$, we have*

$$\iota_{XA} = \iota_{XB} \circ \iota_{BA} \colon A \to X,$$

i.e. cancellation in the middle.

We make first some observations on a pull-back induced by an inclusion.

Observation 1.5.2. *Let $A \subset X$ be a subset. Then the pull-back $\iota^! \colon \Phi^k(X) \to \Phi^!(A)$ inclusion map $\iota \colon A \hookrightarrow X$ is the restriction map*

$$\iota^! \colon \phi \mapsto \phi|_{A^{k+1}}.$$

Lemma 1.5.3. *Let $U \subset X$ be an open subset. Then*

$$\mathrm{spt}(\iota^!_{XU}\phi) = \mathrm{spt}(\phi) \cap U.$$

for every $\phi \in \Phi^k(X)$. In particular,

$$\mathrm{spt}(\iota^{\#}_{XU} c) = \mathrm{spt}(c) \cap U$$

for each $c \in C^k(X)$.

Proof. We show that $\mathrm{null}(\iota^!_{XU}\phi) = \mathrm{null}(\phi) \cap U$. Clearly, $\mathrm{spt}(\iota^!_{XU}\phi) \subset \mathrm{spt}(\phi) \cap U$. Let now $x \in \mathrm{null}(\iota^!\phi)$. Then there exists a neighborhood W of x in U so that $(\iota^!\phi)|_{W^{k+1}} = 0$. Since W is open in X, we have that

$$\phi|_{W^{k+1}} = (\iota^!\phi)|_{W^{k+1}} = 0.$$

Thus $x \in U \cap \mathrm{null}(\phi)$ and $\mathrm{null}(\iota^!_{XU}\phi) = \mathrm{null}(\phi) \cap U$. The second claim follows immediately. $\square$

1.5.1 Push-forward of k-functions

As mentioned in the introduction to this section, we defined the push-forward of compactly supported k-functions using auxiliary neighborhoods of supports. Let $U \subset X$ be an open set and let

$$R^k_c(X, U) = \{\phi \in \Phi^k_c(X) \colon \mathrm{spt}(\phi) \subset U\}.$$

Remark 1.5.4. *The reader may wonder the name $R^k_c(X,U)$ instead of the more natural $\Phi^k_c(X,U)$. We follow here the naming convention in Massey [Mas78], which is followed also in followed also in the forthcoming sections, where we denote $\Phi^k_c(X,U)$ the kernel of the $(\iota_{XU})^!$ in $\Phi^k_c(X)$.*

Note that,

$$\Phi^k_c(X) = \bigcup_U R^k_c(X,U),$$

where U ranges over the open subsets of X. Note also that, clearly, the sets $R^k_c(X,U)$ are not disjoint and that, given a k-function $\phi \in \Phi^k_c(X)$ there no canonical choice for the neighborhood U of the support of ϕ. Finally, note that $R^k_c(X,U)$ is a subgroup of $\Phi^k_c(X)$.

Definition 1.5.5. Let U and V be open sets in X for which $\overline{V} \subset U$. The homomorphism $(\iota_{XU})^V_! : R^k_c(U,V) \to \Phi^k(X)$ is defined by

$$(\iota_{XU})^V_! \phi(x) = \begin{cases} \chi_{V^{k+1}}(x)\phi(x), & x \in U^{k+1} \\ 0, & \text{otherwise} \end{cases}$$

for $\phi \in R^k_c(U,V)$.

The reason for the auxiliary set V is that the mere zero extension $\Phi^k_c(U) \to \Phi^k_c(X)$ does not resepect the support of k-functions.[3].

Example 1.5.6. *Let $X = \mathbb{R}$, $U = \mathbb{R} \setminus \{0\}$, and let $E \colon \Phi^k_c(X) \to \Phi^k(X)$ be the zero extension of k-functions, that is,*

$$E\phi(x) = \begin{cases} \phi(x), & x \in U^{k+1} \\ 0, & \textit{otherwise} \end{cases}$$

for $\phi \in \Phi^k_c(U$.

Consider now the 0-function $F \colon U \to \mathbb{Z}_+$, $x \mapsto x/|x|$. Then $\delta F(x_1,x_2) = F(x_2) - F(x_1) \neq 0$ for $x_1x_2 < 0$ and $\delta F(x_1,x_2) = 0$ for $x_1x_2 > 0$. In particular, $\operatorname{spt}(\delta F) = \emptyset$.

Then $\operatorname{spt}(E(\delta F)) = \{0\}$. Hence $E\Phi^0_0(U) \not\subset \Phi^0_0(X)$. Moreover, for $x > 0$,

$$\begin{aligned} \delta E(\delta F)(-x,x,0) &= E(\delta F)(x,0) - (E\delta F)(-x,0) + E(\delta F)(-x,x) \\ &= \delta F(-x,x) = F(x) - F(-x) = 2. \end{aligned}$$

Thus $\delta E(\delta F) \neq E(\delta\delta F) = 0$.

It is important to notice that the push-forward $(\iota_{XU})^V_!$ does not commute with the coboundary δ as the following example reveals.

[3]We thank Toni Annala and Eerik Norvio for pointing out mistakes in the earlier version and for suggestions.

Example 1.5.7. *Let $X = \mathbb{R}$, $U = \mathbb{R} \setminus \{0\}$, and $V = \mathbb{R} \setminus [-1, 1]$. Let also $F \colon U \to Z_+$ be the function $F = \chi_{[2,\infty)}$.*

Let $x \geq 2$. Then

$$(\iota_{XU})_!^V \delta F(0, x) = \chi_{V^2}(0, x)\delta F(0, x) = 0.$$

On the other hand,

$$\delta(\iota_{XU})_!^V F(0, x) = (\iota_{XU})_!^V F(x) - (\iota_{XU})_!^V F(0) = \chi_V(x)F(x) - 0 = F(x) = 1.$$

Thus $\delta(\iota_{XU})_!^V \neq (\iota_{XU})_!^V \delta$.

In what follows, we show that – on the level of cochains – the push-forward does not depend on the set V and the commutativity holds. The first result is that the push-foward operator $(\iota_{XU})_!^V$ preserves the support of k-functions.

Lemma 1.5.8. *Let $V \subset U$ be open sets in X satisfying $\overline{V} \subset U$, and $\phi \in R_c^k(U, V)$. Then $\operatorname{spt}(\iota_{XU})_!^V \phi = \operatorname{spt} \phi$. In particular, $(\iota_{XU})_!^V \Phi_0^k(U) \subset \Phi_0^k(X)$.*

Proof. We show first that $\operatorname{spt}((\iota_{XU})_!^V \phi) \subset \operatorname{spt} \phi$. Let $x \in U \setminus \operatorname{spt}(\phi)$. Then, in particular, $x \in \operatorname{null}(\phi)$. Hence there exists a neighborhood W of x in U for which $\phi|_{W^{k+1}} = 0$. Then

$$(\iota_{XU})_!^V \phi|_{W^{k+1}} = \chi_{(V \cap W)^{k+1}} \phi|_{W^{k+1}} = 0.$$

Since U is open in X, we have that W is open in X. Thus $x \in \operatorname{null}((\iota_{XU})_!^V \phi)$, and $\operatorname{spt}((\iota_{XU})_!^V \phi) \subset \operatorname{spt}(\phi)$.

Suppose now that $x \in X \setminus U$. Then $W = X \setminus \overline{V}$ is a neighborhood of x in X. Since $W^{k+1} \cap V^{k+1} = \emptyset$, we have $(\iota_{XU}^V)_! \phi|_{W^{k+1}} = 0$. Thus $x \in \operatorname{null}((\iota_{XU}^V)_! \phi)$ also in this case, and $\operatorname{spt}((\iota_{XU}^V)_! \phi) \subset \operatorname{spt}(\phi)$.

To show that $\operatorname{spt}(\phi) \subset \operatorname{spt}((\iota_{XU}^V)_! \phi)$, let $x \in \operatorname{spt}(\phi)$ and W a neighborhood of x in X. Since V is a neighborhood of $\operatorname{spt}(\phi)$, we have that $W \cap V$ is a neighborhood of x in X, and hence also in U. Since $x \in \operatorname{spt}(\phi)$, we have that $\phi|_{(W \cap V)^{k+1}} \neq 0$. Thus

$$(\iota_{XU}^V)_! \phi|_{(W \cap V)^{k+1}} = \chi_{V^{k+1}}|_{(W \cap V)^{k+1}} \phi|_{(W \cap V)^{k+1}} = \phi|_{(W \cap V)^{k+1}} \neq 0.$$

Thus $x \in \operatorname{spt}\left((\iota_{XU}^V)_! \phi\right)$. □

1.5.2 Push-forward of cochains

In spirit of the previous section, let $U \subset X$ be an open set and let

$$Q_c^k(X, U) = \{c \in C_c^k(X) \colon \operatorname{spt}(c)\}.$$

Note that now $Q_c^k(X,U)$ is the image of $R_c^k(X,U)$ under the quotient map $\Phi^k(X) \to C^k(X)$, that is,

$$Q_c^k(X,U) = \{[\phi] \colon C^k(X) \colon \phi \in R_c^k(X,U)\}.$$

Thus $Q_c^k(X,U)$ is a subgroup of $C_c^k(X)$ and $C_c^k(X)$ a union of the sets $Q_c^k(X,U)$ when U ranges over all open sets in X.

By Lemma 1.5.8, homomorphisms $(\iota_{XU})_!^V \colon R_c^k(X,U) \to R_c^k(X,V)$ decend to homomorphisms $(\iota_{XU})_\#^V \colon Q_c^k(X,U) \to Q_c^k(X,V)$ for which the diagrams

$$\begin{array}{ccc} R_c^k(U,V) & \xrightarrow{(\iota_{XU})_!^V} & R_c^k(X,V) \\ \big\downarrow{\scriptstyle \phi \mapsto [\phi]} & & \big\downarrow{\scriptstyle \phi \mapsto [\phi]} \\ Q_c^k(U,V) & \xrightarrow{(\iota_{XU})_\#^V} & Q_c^k(X,V) \end{array}$$

commute.

Lemma 1.5.9. *Let U and V be an open sets in X for which $\overline{V} \subset U$. Then*

1. *for each $c \in Q_c^k(U,V)$, $(\iota_{XU})^\#(\iota_{XU})_\#^V c = c$, and*
2. *for each $c \in Q_c^k(X,V)$, $(\iota_{XU})_\#^V(\iota_{XU})^\# c = c$.*

Proof. For the first claim it suffices to show that, given $\phi \in \Phi_c^k(U,V)$, we have that $\phi - (\iota_{XU})^!(\iota_{XU})_!^V\phi \in \Phi_0^k(U)$. Let $x \in U^{k+1}$. Then

$$\phi(x) - (\iota_{XU})^!(\iota_{XU})_!^V\phi(x) = \phi(x) - \chi_{V^{k+1}}(x)\phi(x) = \begin{cases} 0, & x \in V^{k+1}, \\ \phi(x), & \text{otherwise} \end{cases}$$

Clearly, $V \subset \mathrm{null}(\phi - (\iota_{XU})^!(\iota_{XU})_!^V\phi)$. Suppose $x \notin V$. The $x \notin \mathrm{spt}(\phi)$. Thus there exists a neighborhood W of x in U for which $\phi|_{W^{k+1}} = 0$. Then, clearly,

$$\left(\phi - (\iota_{XU})^!(\iota_{XU})_!^V\phi\right)|_{W^{k+1}} = 0.$$

Thus $X \backslash V \subset \mathrm{null}(\phi - (\iota_{XU})^!(\iota_{XU})_!^V\phi)$. We conclude that $\phi - (\iota_{XU})^!(\iota_{XU})_!^V\phi \in \Phi_0^k(U)$. This proves the first claim.

For the second claim, it suffices to show that $\phi - (\iota_{XU})_!^V(\iota_{XU})^!\phi \in \Phi_0^k(X)$. Let $x \in X^{k+1}$. Then

$$\phi - (\iota_{XU})_!^V(\iota_{XU})^!\phi = \phi - \chi_{V^{k+1}}\phi.$$

An analogous argument as above now yields the claim. $\square$

Corollary 1.5.10. *Let U, V, and W be open sets in X for which $\overline{V} \subset U$ and $\overline{W} \subset U$. Then, for each $c \in Q_c^k(U,V) \cap Q_c^k(U,W)$,*

$$(\iota_{XU})_\#^V c = (\iota_{XU})_\#^W c.$$

Proof. Let $c \in Q_c^k(U,V) \cap Q_c^k(U,W)$. Since $Q_c^k(U,V) \cap Q_c^k(U,W) = Q_c^k(U, V \cap W)$, we have, by Lemma 1.5.9,

$$\begin{aligned}
(\iota_{XU})_\#^V c &= (\iota_{XU})_\#^V \left((\iota_{XU})^\# (\iota_{XU})_\#^{V\cap W} c \right) \\
&= \left((\iota_{XU})_\#^V (\iota_{XU})^\# \right) (\iota_{XU})_\#^{V\cap W} c = (\iota_{XU})_\#^{V\cap W} c.
\end{aligned}$$

Similarly, $(\iota_{XU})_\#^W c = (\iota_{XU})_\#^{V\cap W} c$. The claim follows. □

Definition 1.5.11. Let U be an open set in X. The homomorphism

$$(\iota_{XU})_\# \colon C_c^k(U) \to C_c^k(X)$$

defined by $(\iota_{XU})_\# c = (\iota_{XU})_\#^V c$ for each $c \in Q_c^k(U,V)$ and open set V satisfying $\overline{V} \subset U$, is the push-forward induced by the inclusion $\iota_{XU} \colon U \hookrightarrow X$.

By Lemma 1.5.9, the push-forward $(\iota_{XU})_\#$ is a right inverse of $(\iota_{XU})^\#$. More precisely, we have the following result. For its importance, we record it as a proposition.

Proposition 1.5.12. *Let U be an open set in X. Then*

$$(\iota_{XU})^\# (\iota_{XU})_\# = \mathrm{id}$$

and

$$(\iota_{XU})_\# (\iota_{XU})^\# |_{Q_c^k(X,U)} = \mathrm{id}.$$

Proof. The first claim follows directly from Lemma 1.5.9. Indeed, let $c \in C_c^k(U)$ and let V be an open set for which $c \in Q_c^k(U,V)$. Then

$$(\iota_{XU})^\# (\iota_{XU})_\# c = (\iota_{XU})^\# (\iota_{XU})_\#^V c = c.$$

Similarly, the second claim follows from Lemma 1.5.9. Indeed, let $c \in Q_c^k(X,U)$. Since $(\iota_{XU})^\# c \in \mathbb{C}_c^k(U)$, there exists an open set V for which $\overline{V} \subset U$ and $(\iota_{XU})^\# c \in C_c^k(U,V)$. Thus

$$(\iota_{XU})_\# (\iota_{XU})^\# c = (\iota_{XU})_\#^V (\iota_{XU})^\# c = c.$$

The claim follows. □

1.5.3 Push-forward in cohomology

It remains to show that the homomorphism $(\iota_{XU})_\# \colon C_c^\#(U) \to C_c^\#(X)$ is a chain map. Although this follows almost immediately from the previous proposition, we record it also as a proposition for its importance.

Proposition 1.5.13. *Let U be an open set in X. Then $d(\iota_{XU})_\# = (\iota_{XU})_\# d$.*

Proof. Let $c \in C_c^k(U)$. Then $d(\iota_{XU})_\# c \in Q_c^{k+1}(X,U)$. Thus, by Proposition 1.5.12,

$$\begin{aligned}(\iota_{XU})_\# dc &= (\iota_{XU})_\# d\Big((\iota_{XU})^\# (\iota_{XU})_\# c\Big) \\ &= (\iota_{XU})_\# (\iota_{XU})^\# d(\iota_{XU})_\# c = d(\iota_{XU})_\# c.\end{aligned}$$

The claim follows. □

The push-forward in cohomology has a special role in the theory. For this reason, we introduce also here the commonly used notation for this operator.

Definition 1.5.14. For an open set $U \subset X$, the homomorphism

$$\tau_{XU} = (\iota_{XU})_* \colon H_c^n(U) \to H_c^n(X)$$

is called the *push-forward (in compactly supported Alexander–Spanier cohomology) induced by the inclusion $U \hookrightarrow X$.*

We note in passing that the composition of inclusions $U \hookrightarrow V$ and $V \hookrightarrow X$ yields the following composition rule.

Observation 1.5.15. *Let $U \subset V \subset X$ be open sets. Then*

$$\tau_{XU} = (\iota_{XU})_* = (\iota_{XV} \circ \iota_{VU})_* = \tau_{XV} \circ \tau_{VU}.$$

Since the push-forward is induced by an inclusion, it is natural to expect that, on the cohomological level, push-forward and pull-back with a proper map commute in a suitable sense. This is indeed the case and we record in the form of the following lemma.

Lemma 1.5.16. *Let $f \colon X \to Y$ be a proper map, $V \subset Y$ a domain, and $U = f^{-1}V$. Then, for each $k \in \mathbb{Z}$, the diagram*

$$\begin{array}{ccc} H_c^k(V) & \xrightarrow{(f|_U)^*} & H_c^k(U) \\ {\scriptstyle \tau_{YV}}\downarrow & & \downarrow{\scriptstyle \tau_{XU}} \\ H_c^k(Y) & \xrightarrow{f^*} & H_c^k(X) \end{array}$$

commutes.

Proof. Recall that $(\iota_{XU})_\# \colon C_c^k(U) \to Q_c^k(X,U)$ and $(\iota_{YV})_\# \colon C_c^k(V) \to Q_c^k(Y,V)$ are isomorphisms, where $Q_c^k(X,U) = \{c \in C_c^k(X) \colon \operatorname{spt}(c) \subset U\}$. The inverses of $(\iota_{XU})_\#$ and $(\iota_{YV})_\#$ are $(\iota_{XU})^\# \colon Q_c^k(X,U) \to C_c^k(U)$ and $(\iota_{YV})^\# \colon Q_c^k(Y,V) \to C_c^k(V)$, respectively.

Since f is proper and $U = f^{-1}V$, we have that $f^{\#}Q^k_c(Y,V) \subset Q^k_c(X,U)$. Thus

$$\begin{aligned} f^{\#} \circ (\iota_{YV})_{\#} &= (\iota_{XU})_{\#} \circ (\iota_{XU})^{\#} \circ f^{\#} \circ (\iota_{YV})_{\#} \\ &= (\iota_{XU})_{\#} \circ (f \circ \iota_{XU})^{\#} \circ (\iota_{YV})_{\#} \\ &= (\iota_{XU})_{\#} \circ (\iota_{VY} \circ f|_U)^{\#} \circ (\iota_{YV})_{\#} \\ &= (\iota_{XU})_{\#} \circ (f|_U)^{\#} \circ (\iota_{YV})^{\#} \circ (\iota_{YV})_{\#} = (\iota_{XU})_{\#} \circ (f|_U)^{\#} \end{aligned}$$

as homomorphisms $C^k_c(V) \to C^k_c(U)$. Hence

$$f^* \circ \tau_{YV} = \tau_{XU} \circ (f|_U)^*.$$

The claim follows. □

1.6 Compact supports of cohomology classes

Since elements of $H^*_c(X)$ are called compactly supported cohomology classes, it is reasonable to consider the meaning of this statement more closely. Since cochains which are coboundaries have non-trivial support, it is easy to get convinced that a cohomology class does not have a well-defined support. It turns out, however, that each cohomology class in $H^*_c(X)$ is compactly contained in a pre-compact open subset of X if X is locally compact.

Lemma 1.6.1. *Let X be a locally compact space and $a \in H^k_c(X)$. Then there exists a pre-compact open subset $U \subset X$ and $b \in H^k_c(U)$ for which $a \in \tau_{XU}(b)$.*

Proof. Let $u \in C^k_c(X)$ be a cochain representing a, that is, $a = [u]$. Since $\operatorname{spt} u$ is well-defined and compact, there exists a pre-compact open set U containing $\operatorname{spt} u$, that is, $\overline{U}$ is compact and $\operatorname{spt} u \subset U$. Since $u \in Q^k_c(X,U)$ and $Q^k_c(X,U) = (\iota_{XU})_{\#}C^k_c(U)$, there exists $v \in C^k_c(U)$ for which $(\iota_{XU})_{\#}(v) = u$. Thus $\tau_{XU}([v]) = [(\iota_{XU})_{\#}(v)] = [u]$. □

We also have the following result which heuristically states that if a cocycle $c \in C^k_c(U)$ is a coboundary in $C^k_c(X)$ then it is coboundary already in $C^k_c(W)$ for some pre-compact open set W. Again, we need to assume that X is locally compact.

Lemma 1.6.2. *Let X be a locally compact space, $U \subset X$ a pre-compact open subset, and $a \in \ker \tau_{XU}$. Then there exists a pre-compact open subset $V \subset X$ containing U so that $a \in \ker \tau_{VU}$.*

Proof. Let $u \in C^k_c(U)$ be a cochain representing a. Since $[(\iota_{XU})_{\#}u] = \tau_{XU}(a) = 0$, there exists $v \in C^{k-1}_c(X)$ for which $(\iota_{XU})_{\#}u = dv$. Let V be a pre-compact open neighborhood of $\mathrm{spt}(u) \cup \mathrm{spt}(v)$. Then $v \in Q^k_c(X, V)$, $(\iota_{XV})^\# v \in C^k_c(V)$, and

$$\begin{aligned} d(\iota_{XV})^\# v &= (\iota_{XV})^\# dv = (\iota_{XV})^\# (\iota_{XU})_\# u \\ &= (\iota_{XV})^\# (\iota_{XV})_\# (\iota_{VU})_\# u = (\iota_{VU})_\# u. \end{aligned}$$

Hence

$$(\iota_{VU})_* a = [(\iota_{VU})_\# u] = [d(\iota_{XV})^\# v] = 0.$$

□

1.7 Cohomology of disconnected spaces

As another application we record again easy but important result on cohomology of disconnected spaces. The two fundamental observations, on level of k-functions and cochains, are the following.

Observation 1.7.1. *Let X be a space, $\mathscr{U} = \{U_i\}_{i\in\Lambda}$ a covering of X with mutualy disjoint open sets, and $\phi \in \Phi^k(X)$. Then ϕ is locally equivalent to the k-function*

$$\phi_{\mathscr{U}} = \sum_{i\in I} \phi|_{U_i^{k+1}} = \sum_{i\in\Lambda} (\iota_{XU_i})_! (\iota_{XU_i})^! \phi.$$

In particular, $[\phi] = [\phi_{\mathscr{U}}]$ as k-cochains in $C^k(X)$. Furthermore, for $c \in C^k_c(X)$,

$$c = \sum_{i\in I} (\iota_{XU_i})_\# (\iota_{XU_i})^\# c.$$

Observation 1.7.2. *Let X be a space, $\mathscr{U} = \{U_i\}_{i\in\Lambda}$ a covering of X with mutualy disjoint open sets, $i \in \Lambda$, and $\phi \in \Phi^k(U_i)$. Then*

$$(\iota_{XU_j})^! (\iota_{XU_i})_! \phi = \begin{cases} \phi, & j = i \\ 0, & j \neq i \end{cases}$$

In particular, for $(c_i)_{i\in\Lambda} \in \bigoplus H^k_c(U_i)$,

$$(\iota_{XU_j})^\# \sum_{i\in\Lambda} (\iota_{XU_i})_\# c_i = c_j$$

for each $j \in \Lambda$.

Theorem 1.7.3. *Let X be a space and $\mathscr{U} = \{U_i\}_{i\in\Lambda}$ a covering of X by mutually disjoint open sets $U_i \subset X$. Then*

$$J \colon \bigoplus_{i\in\Lambda} H^k_c(U_i) \to H^k_c(X), \quad (c_i)_{i\in\Lambda} \mapsto \sum_{i\in\Lambda} \tau_{XU_i} c_i,$$

is an isomorphism and

$$I\colon H^k_c(X) \to \bigoplus_{i\in\Lambda} H^k_c(U_i), \quad c \mapsto ((\iota_{XU_i})^* c)_{i\in\Lambda}.$$

is its inverse.

Proof. To check that J is well-defined, it suffices to note that there are finitely many non-zero terms in the sequence $(c_i)_{i\in\Lambda}$. Clearly, J is a homomorphism.

To show that I is well-defined, let $[c] \in H^k_c(X)$. For each $i \in \Lambda$, the set U_i is both open and closed, and hence ι_{XU_i} is proper. Since $\mathrm{spt}(c)$ is compact, there exists finitely many $i \in \Lambda$ for which $\mathrm{spt}(c) \cap U_i \neq \emptyset$. Thus I is well-defined. Clearly, I is a homomorphism.

Let $[c] \in H^k_c(X)$. Then

$$(J \circ I)[c] = \sum_{i\in\Lambda} \tau_{XU_i}(\iota_{XU_i})^*[c] = \left[\sum_{i\in\Lambda}(\iota_{XU_i})_\#(\iota_{XU_i})^\# c\right] = [c].$$

Let $([c_i])_{i\in\Lambda} \in \bigoplus_{i\in\Lambda} H^k_c(U_i)$. Then

$$\begin{aligned}(I \circ J)([c_i])_{i\in\Lambda} &= \left((\iota_{XU_j})^* \sum_{i\in\Lambda} \tau_{XU_i}[c_i]\right)_{j\in\Lambda} \\ &= \left(\left[(\iota_{XU_j})^\# \sum_{i\in\Lambda}(\iota_{XU_i})_\# c_i\right]\right)_{j\in\Lambda} = ([c_j])_{j\in\Lambda}.\end{aligned}$$

Thus I is the inverse of J, and J is an isomorphism. □

1.8 Retraction of the support

In this section we prove a result which apprears rather techinical at the first glance but turns out to be an important ingredient in the proof of the long exact sequence for a pair.

Heuristically, the result we prove states that, given a k-function which is locally trivial over a closed subset, we may retract a support away from A using a coboundary. We formulate now the result more formally. Let, from now on in this section, X be a locally compact and second countable, Hausdorff space.

We denote

$$\Phi^k_c(X, A) = \{\phi \in \Phi^k_c(X)\colon \phi|_{A^{k+1}} \text{ is locally trivial}\}$$

for each $k \in \mathbb{Z}$ and

$$\Phi^!_c(X, A) = \left(\Phi^k_c(X, A), \delta\right)_{k\in\mathbb{Z}}$$

the corresponding complex. Note that $R^!_c(X, X \setminus A)$ is a subcomplex of $\Phi^!_c(X, A)$. Indeed, if $\mathrm{spt}(\phi) \subset X \setminus A$, then $A \subset \mathrm{null}(\phi)$ and $\phi|_{A^{k+1}}$ is locally trivial as a k-function in $\Phi^k_c(X, A)$.

Note also that $\phi|_{A^{k+1}}$ is locally trivial if and only if $\mathrm{spt}((\iota_{AX})^! \phi) = \emptyset$.

Theorem 1.8.1. *Let X be locally compact, second countable, and Hausdorff and let $A \subset X$ be a closed subset. Let $\phi \in \Phi^k_c(X, A)$ be a k-function for which $\delta\phi \in R^{k+1}_c(X, X \setminus A)$. Then there exists $\psi \in R^k_c(X, X \setminus A)$ and $\rho \in \Phi^{k-1}_c(X, A)$ for which*

$$\phi = \psi + \delta\rho.$$

Before moving to the proof of this theorem, we record its consequence on the level of cochains.

Let

$$C^k_c(X, A) = \ker\Big((\iota_{XA})^\# \colon C^k_c(X) \to C^k_c(A)\Big)$$

for each $k \in \mathbb{Z}$, and let again

$$C^\#_c(X, A) = \Big(C^k_c(X, A), d\Big)_{k\in\mathbb{Z}}$$

be the corresponding complex. Clearly, $C^\#_c(X, A)$ is the image of $\Phi^!_c(X, A)$ under the quotient map $\Phi^!_c(X) \to C^\#_c(X)$. Since $Q^\#_c(X; U)$ is the image of $R^!_c(X, U)$ under the quotient map $\Phi^!_c(X) \to C^\#_c(X)$, we have, in particular, that $Q^\#_c(X, X \setminus A)$ is a subcomplex of $C^\#_c(X, A)$.

Theorem 1.8.2. *Let X be a locally compact, second countable, and Hausdorff space, and let $A \subset X$ be a closed set. Then the inclusion $i \colon Q^\#_c(X, X \setminus A) \hookrightarrow C^\#_c(X, A)$ induces an isomorphism*

$$i_* \colon H_k(Q^\#_c(X, X \setminus A)) \to H_k(C^\#_c(X, A)).$$

Proof. For injectivity, suppose a cycle $[\phi] \in Q^k_c(X, X \setminus A)$ is a boundary in $C^\#_c(X, A)$. Then there exists $[\phi'] \in C^{k-1}_c(X, A)$ for which $[\phi] = d[\phi'] = [\delta\phi]$. Thus ψ and $\delta\phi$ are locally equivalent and $\phi = \delta\phi + \beta$, where $\beta \in \Phi^k_0(X)$. In particular, $\delta\phi \in R^k_c(X, X \setminus A)$.

Since $\phi' \in \Phi^k_c(X, A)$ and $\delta\phi \in R^{k+1}_c(X, X \setminus A)$, there exist, by Theorem 1.8.1, $\psi' \in R^k_c(X, X \setminus A)$ and $\rho' \in \Phi^{k-1}_c(X, A)$ for which $\phi' = \psi' + \delta\rho'$. Hence

$$\phi = \delta(\psi' + \delta\rho') + \beta = \delta\psi' + \beta.$$

We conclude that $[\phi] = d[\psi']$, where $[\psi'] \in Q^{k-1}_c(X, X \setminus A)$. Thus $[\psi]$ is a boundary in $Q^\#_c(X, X \setminus A)$. This proves the injectivity.

For surjectivity, let $[\phi] \in C^k_c(X, A)$ be a cycle, that is, $d[\phi] = 0$. Then $\delta\phi$ is locally trivial, and hence $\delta\phi \in R^k_c(X, X \setminus A)$. Since $\phi \in \Phi^k_c(X, A)$ and

$\delta\phi \in R^k_c(X, X \setminus A)$, there exist, by Theorem 1.8.1, $\psi \in R^k_c(X, X \setminus A)$ and $\rho \in \Phi^{k-1}_c(X, A)$ for which $\phi = \psi + \delta\rho$. Thus

$$\delta\psi = \delta(\delta\rho - \phi) = -\delta\phi.$$

Thus

$$d[\psi] = [\delta\psi] = [-\delta\phi] = -d[\phi] = 0.$$

We conclude that $[\psi]$ is a cycle in $Q^k_c(X, X \setminus A)$. Since $[\rho] \in C^\#_c(X, A)$ and $[\phi] = [\psi] + d[\rho]$, we have that $[\phi]$ and $[\psi]$ represent the same homology class in $H_k(C^\#_c(X, A))$. This proves the surjectivity. □

The proof of Theorem 1.8.1 is based on choice of a perturbation of the identity $X \to X$ and a related chain homotopy operator on the level of k-functions. We begin with these preliminaries and the proceed to the proof of the theorem.

1.8.1 Chain homotopy

We begin by showing that an induced homomorphism $f^! \colon \Phi^k(X) \to \Phi^k(X)$ is chain homotopic to the identity. We introduce the following notations. Let $f \colon X \to X$ be a mapping (not necessarily continuous). For each $k \in \mathbb{N}$ and $i = 1, \ldots, k$, we denote

$$F^f_i \colon X^k \to X^{k+1}, \quad (x_1, \ldots, x_k) \mapsto (f(x_1), \ldots, f(x_i), x_i, \ldots x_k)$$

and

$$D_f \colon \Phi^k(X) \to \Phi^{k-1}(X), \quad \phi \mapsto \sum_{i=1}^k (-1)^{i+1} \phi \circ F^f_i;$$

for $k < 0$, we set $D_f = 0$ for completeness.

In particular, for $\phi \in \Phi^k(X)$,

$$(D_f\phi)(x_1, \ldots, x_k) = \sum_{i=1}^k (-1)^{i+1} \phi(f(x_1), \ldots, f(x_i), x_i, \ldots, x_k)$$

for $(x_1, \ldots, x_k) \in X^k$.

The main proposition is that the mapping $D_f \colon \Phi^!(X) \to \Phi^!(X)$ is a chain homotopy.

Proposition 1.8.3. *Let $f \colon X \to X$ be a (non-continuous) map. Then, for each $k \in \mathbb{Z}$,*

$$\mathrm{id} - f^! = \delta D_f + D_F\delta \colon \Phi^k(X) \to \Phi^k(X).$$

Proof. Let $\phi \in \Phi^k(X)$ and $x_1, \ldots, x_{k+1} \in X$. Then

$$(\delta D_f \phi)(x_1, \ldots, x_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} (D_f \phi)(x_1, \ldots, \widehat{x_j}, \ldots, x_{k+1})$$

and

$$(D_f \delta\phi)(x_1, \ldots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (\delta\phi)(f(x_1), \ldots, f(x_i), x_i, \ldots, x_{k+1}).$$

On the other hand, for each $j = 1, \ldots, k+1$,

$$\begin{aligned}
&(D_f \phi)(x_1, \ldots, \widehat{x_j}, \ldots, x_{k+1}) \\
&= \sum_{i=1}^{j-1} (-1)^{i+1} \phi(f(x_1), \ldots, f(x_i), x_i, \ldots, \widehat{x_j}, \ldots, x_{k+1}) \\
&\quad + \sum_{i=j+1}^{k+1} (-1)^{(i+1)-1} \phi(f(x_1), \ldots, \widehat{f(x_j)}, \ldots, f(x_i), x_i, \ldots, x_{k+1})
\end{aligned}$$

and

$$\begin{aligned}
&(\delta\phi)(f(x_1), \ldots, f(x_i), x_i, \ldots, x_{k+1}) \\
&= \sum_{j=1}^{i} (-1)^{j+1} \phi(f(x_1), \ldots, \widehat{f(x_j)}, \ldots, f(x_i), x_i, \ldots, x_{k+1}) \\
&\quad + \sum_{j=i}^{k+1} (-1)^{(j+1)+1} \phi(f(x_1), \ldots, f(x_i), x_i, \ldots, \widehat{x_j}, \ldots, x_{k+1})
\end{aligned}$$

Thus

$$\begin{aligned}
&(\delta D_f \phi)(x_1, \ldots, x_{k+1}) + (D_f \delta\phi)(x_1, \ldots, x_{k+1}) \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i=1}^{j-1} (-1)^{i+1} \phi(f(x_1), \ldots, f(x_i), x_i, \ldots, \widehat{x_j}, \ldots, x_{k+1}) \\
&\quad + \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i=j+1}^{k+1} (-1)^{i} \phi(f(x_1), \ldots, \widehat{f(x_j)}, \ldots, f(x_i), x_i, \ldots, x_{k+1}) \\
&\quad + \sum_{i=1}^{k+1} (-1)^{i+1} \sum_{j=1}^{i} (-1)^{j+1} \phi(f(x_1), \ldots, \widehat{f(x_j)}, \ldots, f(x_i), x_i, \ldots, x_{k+1}) \\
&\quad + \sum_{i=1}^{k+1} (-1)^{i+1} \sum_{j=i}^{k+1} (-1)^{j} \phi(f(x_1), \ldots, f(x_i), x_i, \ldots, \widehat{x_j}, \ldots, x_{k+1})
\end{aligned}$$

By rearranging the double sums, we obtain

$$\begin{aligned}
&(\delta D_f\phi)(x_1,\dots,x_{k+1}) + (D_f\delta\phi)(x_1,\dots,x_{k+1})\\
&\quad=\sum_{i<j}(-1)^{i+j}\phi(f(x_1),\dots,f(x_i),x_i,\dots,\widehat{x_j},\dots,x_{k+1})\\
&\qquad-\sum_{i>j}(-1)^{i+j}\phi(f(x_1),\dots,\widehat{f(x_j)},\dots,f(x_i),x_i,\dots,x_{k+1})\\
&\qquad+\sum_{j\le i}(-1)^{i+j}\phi(f(x_1),\dots,\widehat{f(x_j)},\dots,f(x_i),x_i,\dots,x_{k+1})\\
&\qquad-\sum_{j\ge i}(-1)^{i+j}\phi(f(x_1),\dots,f(x_i),x_i,\dots,\widehat{x_j},\dots,x_{k+1})\\
&\quad=\sum_{i=1}^{k+1}\phi(f(x_1),\dots,f(x_{i-1}),x_i,\dots,x_{k+1})\\
&\qquad-\sum_{i=1}^{k+1}\phi(f(x_1),\dots,f(x_i),x_{i+1},\dots,x_{k+1})\\
&\quad=\phi(x_1,\dots,x_{k+1})-\phi(f(x_1),\dots,f(x_{k+1}))\\
&\quad=\Big((\mathrm{id}-f^!)\phi\Big)(x_1,\dots,x_{k+1}).
\end{aligned}$$

This completes the proof. □

1.8.2 Small perturbation of the identity

We move now to the second tool in the proof of Theorem 1.8.1 – small perturbation of the identity. We recall some terminology related to coverings.

Recall that a covering $\mathscr{V}$ *refines covering* $\mathscr{U}$ if for each $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ for which $V \subset U$, and that a covering $\mathscr{U}$ of X is *locally finite* if for each $x \in X$ there exists a neighborhood W for which $\#\{U \in \mathscr{U} : U \cap W \neq \emptyset\} < \infty$.

A Hausdorff space is *paracompact* if each open covering has a locally finite refinement. Our space X is paracompact.

Fact 1.8.4. *Locally compact, second countable, and Hausdorff spaces are paracompact.*

Indeed, a locally compact, second countable, and Hausdorff space is both regular [Dug78, Theorem XI.6.4] and Lindelöf [Dug78, Theorem XI.7.2]. *A fortiori*, regular Lindelöf spaces are paracompact [Dug78, Theorem VIII.6.5].

The reason to emphasize paracompactness are the star refinements; see [Dug78, Theorem VIII.3.5]. Given a covering $\mathscr{U}$ of X, a *star of* $U \in \mathscr{U}$ *in* $\mathscr{U}$ is the set $U^* = \bigcup\{U' \in \mathscr{U} : U' \cap U \neq \emptyset\}$. A *star* $\mathscr{U}^*$ *of* $\mathscr{U}$ is the covering $\mathscr{U}^* = \{U^* : U \in \mathscr{U}\}$.

Fact 1.8.5. *Each open covering of a paracompact space has a locally finite star refinement, that is, for each open covering $\mathscr{U}$ of X there exists a covering $\mathscr{V}$ for which $\mathscr{V}^*$ is a locally finite refinement of $\mathscr{U}$.*

To define (non-continuous) mappings $f\colon X \to X$, we consider as small perturbations of the identity $\mathrm{id}\colon X \to X$, we need some auxiliary notations.

Let $\mathscr{W}$ be a covering of X and $A \subset X$ a closed subset. We denote $\mathscr{W}_A = \{W \in \mathscr{W} : W \cap A \neq \emptyset\}$. Also, given any subcollection $\mathscr{W}' \subset \mathscr{W}$, the notation $(\mathscr{W}')^*$ refers to the collection $\{W^* \subset X : W' \in \mathscr{W}'\}$, where W^* is the star of W in $\mathscr{W}$.

Definition 1.8.6. Let $\mathscr{W}$ be a covering of the space X and $A \subset X$ a closed subset. A mapping $f\colon X \to X$ is an $(\mathscr{W}, A)$*-perturbation (of the identity)* if

1. $f|_{A \cup X \setminus \bigcup \mathscr{W}_A} = \mathrm{id}$, and
2. for each $W \in \mathscr{W}_A$, $fW \subset W^* \cap A$, where W^* is the star of W in $\mathscr{W}$.

Each covering of X and each closed set in X there exists a perturbation of the identity.

Lemma 1.8.7. *Let $\mathscr{W}$ be a covering of X and $A \subset X$. Then there exists a $(\mathscr{W}, A)$-perturbation $f\colon X \to X$.*

Proof. First, for each $W \in \mathscr{W}_A$, let $x_W \in W \cap A$, and for each $x \in X$, let $W_x \in \mathscr{W}$ be a neighborhood of x. We define now f by

$$x \mapsto \begin{cases} x_{W_x}, & x \in (\bigcup \mathscr{W}_A) \setminus A, \\ x, & \text{otherwise} \end{cases}$$

Then, clearly, $f|_A = \mathrm{id}$ and $f|_{X \setminus \bigcup \mathscr{W}_A} = \mathrm{id}$.

Let now $W \in \mathscr{W}_A$ and $x \in W$. Then $W_x \cap W \neq \emptyset$ and hence $W_x \subset W^*$, where W^* is the star of W in $\mathscr{W}$. Since $f(x) \in W_x$, we conclude that $fW \subset W^*$. Thus the mapping f is a $(\mathscr{W}, A)$-perturbation of the identity. □

We record now basic properties for a perturbation of the identity $f\colon X \to X$ and the associated chain homotopies $D_f\colon \Phi^!(X) \to \Phi^!(X)$ in the case of locally finite coverings having pre-compact elements. We begin a statement on the supports, which holds without additional assuptions on covers.

In the following lemmas, delicate considitions regarding the covering $\mathscr{W}$ are imposed to the k-functions in $\Phi^k(X)$. These conditions are necessary as the following example reveals.

Example 1.8.8. *Let $X = \mathbb{R}$ and $A = \mathbb{Z}$. Let also $\mathscr{W} = \{(x-1, x+1) : x \in \mathbb{Z}\}$. Let $\phi \in \Phi^1(X)$ be the 1-function satisfying $\phi(x, x-1) = 1$ for each $x \in \mathbb{Z}$ and $\phi(x, y) = 0$ otherwise. Then $\mathrm{spt}(\phi) = \emptyset$. Let also $f\colon X \to X$ be the map for which $f(x)$ be the integer part of $x \in \mathbb{R}$. Then f is a $(\mathscr{W}, A)$-perturbation and $\mathrm{spt}(f^!\phi) = \mathbb{Z}$. In particular, $f^!\phi$ is not compactly supported.*

Lemma 1.8.9. *Let $\mathscr{W}$ be an open covering of X, $A \subset X$ a closed subset, and $f\colon X \to X$ a $(\mathscr{W}, A)$-perturbation. Let also $\phi \in \Phi^k(X)$ be a k-function for which $\phi|_{(W^* \cap A)^{k+1}} = 0$ for each $W \in \mathscr{W}_A$. Then* $\mathrm{spt}(f^!\phi) \subset \mathrm{cl}\left(\bigcup \mathscr{W}^*_{\mathrm{spt}\,\phi}\right) \setminus A$.

Proof. We show first that $A \subset \mathrm{null}(f^!\phi)$. Let $x \in A$ and let $W \in \mathscr{W}$ be an element containing x. Let also $x_1, \ldots, x_{k+1} \in W$. Since $W \in \mathscr{W}_A$, we have that $(f(x_1), \ldots, f(x_{k+1})) \in (W^*)^{k+1}$. Thus

$$f^!\phi(x_1, \ldots, x_{k+1}) = \phi(f(x_1), \ldots, f(x_{k+1})) = 0.$$

Hence $f^!\phi|_{(W^*)^{k+1}} = 0$ and $x \in \mathrm{null}(f^!\phi)$.

It remains to show that $\mathrm{spt}(f^!\phi) \subset \mathrm{cl}\bigcup \mathscr{W}^*_{\mathrm{spt}\,\phi}$. Let $x \notin \mathrm{cl}\bigcup \mathscr{W}^*_{\mathrm{spt}\,\phi}$ and $W \in \mathscr{W}$ a neighborhood of x.

Suppose first that $W \in \mathscr{W}_A$. Let $V = W \setminus \mathrm{cl}\bigcup \mathscr{W}^*_\phi$ and $(x_1, \ldots, x_{k+1}) \in V^{k+1}$. Then $(f(x_1), \ldots, f(x_{k+1})) \in (W^*)^{k+1} \cap A^{k+1}$ and

$$f^!\phi(x_1, \ldots, x_{k+1}) = \phi(f(x_1), \ldots, f(x_{k+1})) = 0$$

by assumption $\phi|_{(W^* \cap A)^{k+1}} = 0$. Hence $x \in \mathrm{null}(f^!\phi)$.

Suppose now that $W \notin \mathscr{W}_A$. We observe that $W \cap \bigcup \mathscr{W}_\phi = \emptyset$. Indeed, otherwise, there exists $W' \in \mathscr{W}_\phi$ for which $W \cap W' \neq \emptyset$ and then $W \subset (W')^* \subset \bigcup \mathscr{W}^*_\phi$, which is a contradiction.

Since $x \notin \mathrm{spt}(\phi)$, there exists a neighborhood V of x contained in W for which $\phi|_{V^{k+1}} = 0$. Thus $f|_W = \mathrm{id}$ and $\phi|_{V^{k+1}} = 0$. Hence

$$f^!\phi|_{V^{k+1}} = \phi|_{V^{k+1}} = 0.$$

Hence $x \in \mathrm{null}(f^!\phi)$. □

Regarding the mapping properties of the chain homotopy D_f we have the following lemmas.

Lemma 1.8.10. *Let $\mathscr{W}$ be an open covering of X, $A \subset X$ a closed subset, $f\colon X \to X$ a $(\mathscr{W}, A)$-perturbation. Let $\psi \in \Phi^k_c(X)$ be a k-function satisfying $\psi|_{(W^*)^{k+1}} = 0$ for each $W \in \mathscr{W}_A$. Then*

$$\mathrm{spt}(D_f\psi) \subset \mathrm{cl}\left(\bigcup \mathscr{W}^*_{\mathrm{spt}\,\psi}\right) \setminus A.$$

Proof. We show first that $A \subset \mathrm{null}(\psi \circ F_i)$ for each $i = 1, \ldots, k$. Let $x \in A$ and let $W \in \mathscr{W}$ be a neighborhood of x in X. Let also $x_1, \ldots, x_k \in W$. Since $W \in \mathscr{W}_A$, we have

$$F_i^f(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_i), x_i, \ldots, x_k) \in (W^*)^i \times W^{k-i+1} \subset (W^*)^{k+1}$$

and

$$(\psi \circ F_i^f)(x_1, \ldots, x_k) = 0.$$

We conclude that $x \in \text{null}(\phi \circ F_i^f)$. Hence $A \subset \text{null}(\phi \circ F_i^f)$ for each $i = 1, \ldots, k$. Thus $\text{spt}(D_f \psi) \subset \bigcup_{i=1}^k \text{spt}(\psi \circ F_i^f) \subset X \setminus A$.

It remains to to show that $\text{spt}(\psi \circ F_i) \subset \text{cl}\left(\bigcup \mathscr{W}^*_{\text{spt}\,\psi}\right)$ for each $i = 1, \ldots, k$.. Let $x \notin \text{cl} \bigcup \mathscr{W}^*_{\text{spt}\,\phi}$ and $W \in \mathscr{W}$ a neighborhood of x. Let also $i \in \{1, \ldots, k\}$.

Suppose first that $W \in \mathscr{W}_A$. Let $V = W \setminus \text{cl} \bigcup \mathscr{W}^*_\phi$ and $(x_1, \ldots, x_k) \in V^k$. Then $(f(x_1), \ldots, f(x_i), x_i, \ldots, x_k) \in (W^*)^{k+1}$ and

$$\phi \circ F_i^f(x_1, \ldots, x_k) = \phi(f(x_1), \ldots, f(x_i), x_i, \ldots, x_k) = 0$$

by the assumption $\phi|_{(W^*)^{k+1}} = 0$. Hence $x \in \text{null}(\phi \circ F_i^f)$.

Suppose now that $W \notin \mathscr{W}_A$. Similarly as in Lemma 1.8.9, we observe that $W \cap \bigcup \mathscr{W}_\phi = \emptyset$. Since $x \notin \text{spt}(\phi)$, there exists a neighborhood V of x contained in W for which $\phi|_{V^{k+1}} = 0$. Thus $f|_W = \text{id}$ and $\phi|_{V^{k+1}} = 0$. Hence

$$\phi \circ F_i^f|_{V^{k+1}} = \phi|_{V^{k+1}} = 0.$$

Hence $x \in \text{null}(\phi \circ F_i^f)$. The claim follows. □

Lemma 1.8.11. *Let $\mathscr{W}$ be a open covering of X, $A \subset X$ a closed subset, $f\colon X \to X$ a $(\mathscr{W}, A)$-perturbation, and $\phi \in \Phi^k(X)$ be a k-function for which $(\iota_{XA})^! \phi$ is locally trivial. Then $(\iota_{XA})^! D_f \phi$ is locally trivial.*

Proof. Let $\phi \in \Phi^k(X)$ be a k-function for which $(\iota_{XA})^! \phi$ is locally trivial. It suffices to show that $\phi \circ F_i^f|_{A^k}$ is locally trivial for each $i = 1, \ldots, k$. Let $x \in A$. Since $\phi|_{A^{k+1}}$ is locally trivial, there exists a neighborhood V of x in A for which $\phi|_{V^{k+1}} = 0$.

Let $i \in \{1, \ldots, k\}$. Since $f|_A = \text{id}$, we have

$$\phi \circ F_i^f|_{V^k} = \phi|_{V^{k+1}} = 0$$

for each $i = 1, \ldots, k$. Thus $D_f \phi|_{V^k} = 0$. Hence $D_f \phi|_{A^k}$ is locally trivial. □

1.8.3 Proof of the retraction theorem (Theorem 1.8.1)

Let $\phi \in \Phi^k_c(X, A)$ be as in the claim, that is, $\phi|_{A^{k+1}} = (\iota_{AX})^! \phi$ is locally trivial and $\delta\phi \in R^{k+1}_c(X, U)$. We begin by constructing a covering $\mathscr{W}$ of X associated to $\phi|_{A^{k+1}}$, ϕ, and $\delta\phi$.

First, for each $x \in A$, we fix a pre-compact neighborhood V_x of x in X for which $\phi|_{(V_x \cap A)^{k+1}} = 0$. Then $\mathscr{V} = \{V_x\}_{x \in A} \cup \{X \setminus A\}$ is an open covering of X. Note that, given $V \in \mathscr{V}_A$, there exists $x \in A$ for which $V = V_x$ and hence $\phi|_{(V \cap A)^{k+1}} = 0$.

Second, for each $x \notin \text{spt}(\delta\phi)$, let V'_x be a neighborhood of x in X for which $\delta\phi|_{(V'_x)^{k+2}} = 0$. Then $\mathscr{V}' = \{V'_x\}_{x \notin \text{spt}(\delta\phi)} \cup \{X \setminus \text{cl}(\text{null}(\delta\phi))\}$ is a covering of X. Again, note that, if $V' \in \mathscr{V}'$ meets $\text{null}(\delta\phi)$, then $V' \subset$

null($\delta\phi$) and there exists $x \in \operatorname{null}(\delta\phi)$ for which $V' = V'_x$. Hence $\delta\phi|_{(V')^{k+2}} = 0$.

Third, for each $x \notin \bigcup \mathscr{W}_{\operatorname{spt}\phi}$, let V''_x be a neighborhood of x in X for which $\phi|_{(V''_x)^{k+1}} = 0$. Then $\mathscr{V}'' = \{V''_x\}_{x \notin \operatorname{spt}\phi} \cup \{\bigcup \mathscr{W}_{\operatorname{spt}\phi}\}$ is a covering of X.

Since X is paracompact, there exists a locally finite cover $\mathscr{W}$ of X which is a simultaneous star refinement of $\mathscr{V}$, $\mathscr{V}'$, and $\mathscr{V}''$, that is, given $W \in \mathscr{W}$ there exist $V \in \mathscr{V}$, $V' \in \mathscr{V}'$, and $V'' \in \mathscr{V}''$ for which $W^* \subset V \cap V' \cap V''$. Let $f\colon X \to X$ be a $(\mathscr{W}, A)$-perturbation.

It suffices to show that

(i) $f^!\phi \in R^k_c(X, U)$,

(ii) $D_f\delta\phi \in R^k_c(X, U)$, and

(iii) $D_f\phi \in \Phi^{k-1}_c(X, A)$.

Indeed, by Proposition 1.8.3,

$$\phi = \left(f^!\phi + D_f\delta\phi\right) + \delta D_f\phi.$$

Thus we may take $\psi = f^!\phi + D_f\delta\phi$ and $\rho = D_f\phi$.

To show (i), let $W \in \mathscr{W}_A$. Then there exists $V \in \mathscr{V}$ for which $W^* \subset V$ and $\phi|_{(V\cap A)^{k+1}} = 0$. Thus $\phi|_{(W^*\cap A)^{k+1}} = 0$. By Lemma 1.8.9, $\operatorname{spt}(f^!\phi) \subset \operatorname{cl}\left(\bigcup \mathscr{W}^*_{\operatorname{spt}\phi}\right) \setminus A$. Since $\operatorname{cl}\left(\bigcup \mathscr{W}^*_{\operatorname{spt}\phi}\right)$ is compact, we have $f^!\phi \in R^k_c(X, U)$.

To prove (ii), let again $W \in \mathscr{W}_A$. Since $\mathscr{W}$ is a star refinement of $\mathscr{V}'$, there exists $V' \in \mathscr{V}'$ containing W. Since $V' \cap A \neq \emptyset$, we conclude that $V' = V'_x$ for some $x \notin \operatorname{spt}(\delta\phi)$. Since $\delta\phi|_{(V')^*} = 0$, we have also $\delta\phi|_{(W^*)^{k+2}} = 0$. Thus $\operatorname{spt}(D_f\delta\phi) \subset \operatorname{cl}\left(\bigcup \mathscr{W}^*_{\operatorname{spt}\delta\phi}\right) \setminus A$ by Lemma 1.8.10. Since $\operatorname{cl}\left(\bigcup \mathscr{W}^*_{\operatorname{spt}\delta\phi}\right)$ is compect, we have $D_f\delta\phi \in R^k_c(X, U)$.

For the proof of (iii), we observe first that, since $(\iota_{XA})^!\phi$ is locally trivial, so is $(\iota_{XA})^! D_f\phi$ by Lemma 1.8.11. Thus to show that $D_f\phi$ has compact support, it suffices to show that $\operatorname{spt}(D_f\phi) \subset \operatorname{cl}\bigcup \mathscr{W}^*_{\operatorname{spt}\phi}$. Let $x \notin \operatorname{cl}\bigcup \mathscr{W}^*_{\operatorname{spt}\phi}$ and $W \in \mathscr{W}$ a neighborhood of x. Then there exists $x' \in\!\!\!/ \bigcup \mathscr{W}_{\operatorname{spt}\phi}$ for which $W^* \subset V''_x$. Since $\phi|_{(V''_x)^{k+1}} = 0$ we conclude that $\phi|_{(W^*)^{k+1}} = 0$. Let $(x_1, \ldots, x_k) \in W^k$. Then, for each $i = 1, \ldots, k$,

$$F^f_i(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_i), x_i, \ldots, x_k) \in (W^*)^{k+1}.$$

Hence $D_f\phi|_{(W^*)^k} = 0$ and $x \in \operatorname{null}(D_f\phi)$. Thus $\operatorname{spt}(D_f\phi) \subset \operatorname{cl}\bigcup \mathscr{W}^*_{\operatorname{spt}\phi}$. Hence $D_f\phi \in \Phi^{k-1}_c(X, A)$. This completes the proof of Theorem 1.8.1.

1.9 Exact sequence of a pair

As discussed in the beginning of Section 1.8, our main interest for Theorem 1.8.1 stems from the proof of the exact sequence of a pair (X, A) for a closed subset $A \subset X$.

Theorem 1.9.1. *Let X be a locally compact and second countable Hausdorff space and $A \subset X$ a closed subset. Then there exists, for each $k \in \mathbb{Z}$, a homomorphism $\partial\colon H_c^k(A) \to H_c^{k+1}(X \setminus A)$ for which the sequence*
(1.9.1)

$$\cdots \longrightarrow H_c^k(X \setminus A) \xrightarrow{\tau_{XU}} H_c^k(X) \xrightarrow{\iota_{XA}^*} H_c^k(A) \xrightarrow{\partial_k} H_c^{k+1}(X \setminus A) \longrightarrow \cdots$$

is exact.

The interesting aspect in the sequence (1.9.1) is that contains only of Alexander–Spanier groups of spaces, and there are no relative groups in the sense that none of the groups is defined as the homology group of a quotient complex. In the proof this is reflected by the fact that there is no (immediate) short exact sequence which implies this long exact sequence.

Structurally it is also an interesting aspect that both the push-forward τ_{XU} and the pull-back ι_{XU}^* are induced by inclusions, here $X \setminus A \hookrightarrow X$ and $A \hookrightarrow X$, respectively.

The connecting homomorphism $\partial_k\colon H_c^n(A) \to H_c^n(X \setminus A)$ in (1.9.1) is defined as follows.

Recall that the pull-back $\iota_{XU}^\#\colon Q_c^\#(X, U) \to C_c^\#(U)$ is an isomorphism of chain complexes. Let $(\iota_{XU})_*\colon H_{k+1}(Q_c^\#(X, U)) \to H_c^{k+1}(U)$ be the isomorphism in homology induced by $\iota_{XU}^\#$; note that $H_c^*(U) = H_*(C_c^\#(U))$. Recall also that, by Theorem 1.8.2, the inclusion $j\colon Q_c^\#(X, U) \to C_c^\#(X, A)$ of complexes induces an isomorphism $j_*\colon H_{k+1}(Q_c^\#(X, U)) \to H_{k+1}(C_c^\#(X, A))$.

Let now $\partial_k'\colon H_c^k(A) \to H_{k+1}(C_c^\#(X, A))$ be the connecting homomorphism in the long exact sequence

$$\longrightarrow H_c^k(X) \longrightarrow H_c^k(A) \xrightarrow{\partial_k'} H_{k+1}(C_c^\#(X, A)) \longrightarrow \tag{1.9.2}$$

induced by the short exact sequence

$$0 \longrightarrow C_c^k(X, A) \hookrightarrow C_c^k(X) \xrightarrow{\iota_{XA}^\#} C_c^k(A) \longrightarrow 0. \tag{1.9.3}$$

The connecting homomorphism $\partial_k\colon H_c^k(A) \to H_c^k(X \setminus A)$ is now the homomorphism for which the diagram

$$\begin{array}{ccc} H_c^k(A) & \overset{\partial_k}{\dashrightarrow} & H_c^{k+1}(U) \\ \big\downarrow{\scriptstyle \partial_k'} & & \big\uparrow{\scriptstyle \cong\ (\iota_{XU})_*} \\ H_{k+1}(C_c^\#(X, A)) & \underset{j_*}{\overset{\cong}{\longleftarrow}} & H_{k+1}(Q_c^\#(X, U)) \end{array}$$

commutes, that is,

$$\partial_k = (\iota_{XU})_* \circ j_*^{-1} \circ \partial'_k. \tag{1.9.4}$$

Proof of Theorem 1.9.1. Since $Q_c^k(X,U)$ is contained in the kernel of $(\iota_{XA})^\#$, we may factorize $(\iota_{XA})^\# \colon C_c^k(X) \to C_c^k(A)$ through $C_c^k(X)/Q_c^k(X,U)$, that is, we have the diagram

$$\begin{array}{ccc} C_c^k(X) & \xrightarrow{(\iota_{XA})^\#} & C_c^k(A) \\ {\scriptstyle q}\downarrow & \nearrow{\scriptstyle \bar\iota} & \\ C_c^k(X)/Q_c^k(X,U) & & \end{array}$$

where $q \colon C_c^k(X) \to C_c^k(X)/Q_c^k(X,U)$ is the natural quotient map.

Thus the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_c^k(X,U) & \xhookrightarrow{j'} & C_c^k(X) & \xrightarrow{q} & C_c^k(X)/Q_c^k(X,U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j} & & \Vert{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \bar\iota} & & \\ 0 & \longrightarrow & C_c^k(X,A) & \hookrightarrow & C_c^k(X) & \xrightarrow{\iota_{XA}^\#} & C_c^k(A) & \longrightarrow & 0 \end{array}$$

where $j \colon Q_c^k(X,U) \to C_c^k(X,A)$ is an inclusion, has exact rows and all squares commute.

By naturality of the long exact sequence in homology, the diagram

(1.9.5)

$$\begin{array}{ccccccccc} \longrightarrow & H_c^k(X) & \longrightarrow & H_k(C_c^\#(X)/Q_c^\#(X,U)) & \xrightarrow{\partial''_k} & H_{k+1}(Q_c^\#(X,U)) & \longrightarrow \\ & \Vert{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \bar\iota_*} & & \cong\downarrow{\scriptstyle j_*} & \\ \longrightarrow & H_c^k(X) & \longrightarrow & H_c^k(A) & \xrightarrow{\partial'_k} & H_{k+1}(C_c^\#(X,A)) & \longrightarrow \end{array}$$

commutes. Since the Alexander–Spanier cohomology $H_c^*(\cdot)$ is the homology of the complex $C_c^\#(\cdot)$, we have $H_c^*(X) = H_*(C_c^\#(X))$ and $H_c^*(A) = H_*(C_c^\#(A))$. Thus, by the Five Lemma, $\bar\iota_*$ is an isomorphism.

We combine now all relationships of $H_c^*(A)$, $H_c^*(U)$ and $H_c^*(X)$ into one diagram

(1.9.6)

$$\begin{array}{ccccccc} H_c^k(U) & \xrightarrow{\tau_{XU}} & H_c^k(X) & \xrightarrow{\iota_{XA}^*} & H_c^k(A) & \xrightarrow{\partial_k} & H_c^{k+1}(U) \\ \cong\uparrow{\scriptstyle (\iota_{XU})_*} & \nearrow{\scriptstyle j'_*} & {\scriptstyle q_*}\downarrow & \nearrow{\scriptstyle \bar\iota_*}\ \cong & \downarrow{\scriptstyle \partial'_k} & \nearrow{\scriptstyle (\iota_{XU})_*\circ j_*^{-1}}\ \cong & \\ H_k(Q_c^\#(X,U)) & & H_k(C_c^\#(X)/Q_c^\#(X,U)) & & H_{k+1}(C_c^\#(X,A)) & & \end{array}$$

where all triangles commute. The top row in (1.9.6) is exact by (1.9.5). This completess the proof of the exactness of (1.9.1). □

Naturality of the long exact sequence of a pair

We discuss now some naturality statements for the exact sequences of a pair. The first one is almost obvious.

Theorem 1.9.2. *Let $A \subset X$ and $B \subset Y$ be closed sets so that there exists a homeomorphism $f\colon X \to Y$ for which $fA = B$.*[4] *Then the diagram*

$$\begin{array}{ccccccc}
H_c^k(X \setminus A;\mathbb{Z}) & \xrightarrow{\tau_A} & H_c^k(X;\mathbb{Z}) & \xrightarrow{\iota_A^*} & H_c^k(A;\mathbb{Z}) & \xrightarrow{\partial_A} & H_c^{k+1}(X \setminus A;\mathbb{Z}) \\
\uparrow{\scriptstyle (f|_{X\setminus A})^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (f|_A)^*} & & \uparrow{\scriptstyle (f_{X\setminus A})^*} \\
H_c^k(Y \setminus B;\mathbb{Z}) & \xrightarrow{\tau_B} & H_c^k(Y;\mathbb{Z}) & \xrightarrow{\iota_B^*} & H_c^k(B;\mathbb{Z}) & \xrightarrow{\partial_B} & H_c^{k+1}(Y \setminus B;\mathbb{Z})
\end{array}$$

where rows are exact sequences of pairs (X, A) and (Y, B), commute.

Proof. The first square commutes by Lemma 1.5.16 and the second by the composition law for the pull-back. Thus it remains to show that the square

$$\begin{array}{ccc}
H_c^k(A;\mathbb{Z}) & \xrightarrow{\partial_A} & H_c^{k+1}(X \setminus A;\mathbb{Z}) \\
\uparrow{\scriptstyle (f|_A)^*} & & \uparrow{\scriptstyle (f_{X\setminus A})^*} \\
H_c^k(B;\mathbb{Z}) & \xrightarrow{\partial_B} & H_c^{k+1}(Y \setminus B;\mathbb{Z})
\end{array}$$

commutes. Since the connecting homomorphism ∂'_k and isomorphisms j_* and $(\iota_{XU})_*$ are natural in the definition of ∂_k the claim follows from the observation that the diagram

$$\begin{array}{ccccc}
H_c^k(A;\mathbb{Z}) & & \xrightarrow{\qquad\qquad\partial_A\qquad\qquad} & & H_c^{k+1}(X \setminus A;\mathbb{Z}) \\
\uparrow & \searrow{\scriptstyle \partial'_k} & & \nearrow{\scriptstyle \cong\ (\iota_{XU})_*} & \uparrow \\
 & H_{k+1}(C_c^{\#}(X,A)) & \xleftarrow[j_*]{\cong} & H_{k+1}(Q_c^{\#}(X,U)) & \\
{\scriptstyle (f|_A)^*} & {\scriptstyle (f^{\#}|_{C_c^{\#}(Y,B)})_*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle (f^{\#}|_{Q_c^{\#}(Y,V)})_*} & {\scriptstyle (f_{X\setminus A})^*} \\
 & H_{k+1}(C_c^{\#}(Y,B)) & \xleftarrow[j_*]{\cong} & H_{k+1}(Q_c^{\#}(Y,V)) & \\
\uparrow & \nearrow{\scriptstyle \partial'_k} & & \searrow{\scriptstyle (\iota_{YV})_*\ \cong} & \uparrow \\
H_c^k(B;\mathbb{Z}) & & \xrightarrow{\qquad\qquad\partial_B\qquad\qquad} & & H_c^{k+1}(Y \setminus B;\mathbb{Z})
\end{array}$$

commutes. □

A typical version of the naturality of the long exact sequence of a pair reads as follows.

[4] Typical terminology is that f is a homeomorphism of pairs $f\colon (X, A) \to (Y, B)$.

Theorem 1.9.3. *Let A and B be closed subsets of X for which $A \subset B$, and let $U = X \setminus A$ and $V = X \setminus B$. Then the diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_c^k(X \setminus B) & \xrightarrow{\tau_{XV}} & H_c^k(X) & \xrightarrow{\iota_{XB}^*} & H_c^k(B) & \xrightarrow{\partial_B} & H_c^{k+1}(X \setminus B) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \tau_{UV}} & & \|{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota_{BA}^*} & & \downarrow{\scriptstyle \tau_{UV}} & \\
\cdots \longrightarrow & H_c^k(X \setminus A) & \xrightarrow{\tau_{XU}} & H_c^k(X) & \xrightarrow{\iota_{XA}^*} & H_c^k(A) & \xrightarrow{\partial_A} & H_c^{k+1}(X \setminus A) & \longrightarrow \cdots
\end{array}$$

where the rows are exact sequences of pairs (X, A) and (X, B), commutes.

Proof of Theorem 1.9.3. The squares

$$\begin{array}{ccccc}
H_c^k(X \setminus B) & \xrightarrow{\tau_{XV}} & H_c^k(X) & \xrightarrow{\iota_{XB}^*} & H_c^k(B) \\
\downarrow{\scriptstyle \tau_{UV}} & & \| & & \downarrow{\scriptstyle \iota_{BA}^*} \\
H_c^k(X \setminus A) & \xrightarrow{\tau_{XU}} & H_c^k(X) & \xrightarrow{\iota_{XA}^*} & H_c^k(A)
\end{array}$$

commute by the composition laws.

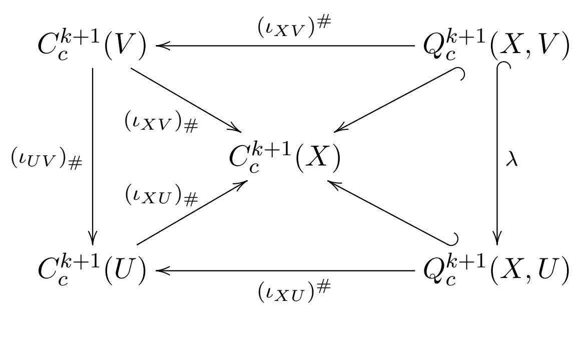
To see that the square

$$\begin{array}{ccc}
H_c^k(B) & \xrightarrow{\partial_B} & H_c^{k+1}(X \setminus B) \\
\downarrow{\scriptstyle \iota_{AB}^*} & & \downarrow{\scriptstyle \tau_{UV}} \\
H_c^k(A) & \xrightarrow{\partial_A} & H_c^{k+1}(X \setminus A)
\end{array}$$

commutes, we begin with an observation that the square

$$\begin{array}{ccc}
Q_c^{k+1}(X, V) & \overset{j_B}{\hookrightarrow} & C_c^{k+1}(X, B) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \kappa} \\
Q_c^{k+1}(X, U) & \overset{j_A}{\hookrightarrow} & C_c^{k+1}(X, A)
\end{array}$$

of inclusions is well-defined and commutes. Similarly, the triangles in the diagram



commute; indeed, the one on left by composition law and the one on the right trivially. Further, we note that the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_c^k(X,B) & \hookrightarrow & C_c^k(X) & \xrightarrow{\iota_{XB}^{\#}} & C_c^k(B) & \longrightarrow & 0 \\ & & \downarrow \kappa & & \| & & \downarrow \iota_{BA}^{\#} & & \\ 0 & \longrightarrow & C_c^k(X,A) & \hookrightarrow & C_c^k(X) & \xrightarrow{\iota_{XA}^{\#}} & C_c^k(A) & \longrightarrow & 0 \end{array}$$

induces a diagram in homology

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_k(C_c^k(X,B)) & \longrightarrow & H_c^k(X) & \longrightarrow & H_c^k(B) & \xrightarrow{\partial'_B} & H_{k+1}(C_c^k(X,B)) & \longrightarrow & \cdots \\ & & \downarrow \kappa_* & & \| & & \downarrow \iota_{BA}^* & & \downarrow \kappa_* & & \\ \cdots & \longrightarrow & H_k(C_c^k(X,A)) & \longrightarrow & H_c^k(X) & \longrightarrow & H_c^k(A) & \xrightarrow{\partial'_A} & H_{k+1}(C_c^k(X,A)) & \longrightarrow & \cdots \end{array}$$

which commutes and has long exact sequences as rows.

Thus the diagram

$$\begin{array}{ccccccc} H_c^{k+1}(V) & \xleftarrow{(i_{XV})_*} & H_{k+1}(Q_c^*(X,V)) & \xrightarrow[\cong]{(j_B)_*} & H_{k+1}(C_c^*(X,B)) & \xleftarrow{\partial'_B} & H_c^k(B) \\ \downarrow \tau_{UV} & & \downarrow \lambda_* & & \downarrow \kappa_* & & \downarrow \iota_{AB}^* \\ H_c^{k+1}(U) & \xleftarrow{(i_{XU})_*} & H_{k+1}(Q_c^*(X,U)) & \xrightarrow[\cong]{(j_A)_*} & H_{k+1}(C_c^*(X,A)) & \xleftarrow{\partial'_A} & H_c^k(A) \end{array}$$

commutes. Since $\partial_A = (i_{XU})_* \circ (j_A)_*^{-1} \circ \partial'_A$ and $\partial_B = (i_{XV})_* \circ (j_B)_*^{-1} \circ \partial'_B$, we have obtained that the square

$$\begin{array}{ccc} H_c^{k+1}(V) & \xleftarrow{\partial_B} & H_c^k(B) \\ \downarrow \tau_{UV} & & \downarrow \iota_{AB}^* \\ H_c^{k+1}(U) & \xleftarrow{\partial_A} & H_c^k(A) \end{array}$$

commutes. This concludes the proof. □

We record one more variant of the naturality of the long exact sequence of a pair for further use.

Theorem 1.9.4. *Let $U \subset X$ be an open set, $W = X \setminus \partial U$, and $A = \partial U$. Then the diagram*

$$\begin{array}{ccccccc} H_c^k(X \setminus \partial U) & \xrightarrow{\tau_{XW}} & H_c^k(X) & \xrightarrow{\iota_{XA}^*} & H_c^k(\partial U) & \xrightarrow{\partial_k^X} & H_c^{k+1}(X \setminus \partial U) \\ \downarrow \iota_{WU}^* & & \downarrow \iota_{X\overline{U}}^* & & \| & & \downarrow \iota_{WU}^* \\ H_c^k(U) & \xrightarrow{\tau_{\overline{U}U}} & H_c^k(\overline{U}) & \xrightarrow{\iota_{\overline{U}A}^*} & H_c^k(\partial U) & \xrightarrow{\partial_k^{\overline{U}}} & H_c^{k+1}(U) \end{array}$$

commutes.

Proof. The first square commutes by Lemma 1.5.16 and the second by the composition law. Thus it suffices to prove that the last square commutes.

By naturality of the long exact sequence (1.9.2), the diagram

$$\begin{array}{ccc} H_c^k(A) & \xrightarrow{\partial'_k} & H_{k+1}(C_c^{\#}(X,A)) \\ \| & & \downarrow{\scriptstyle (\iota^{\#}_{X\overline{U}})_*} \\ H_c^k(A) & \xrightarrow{\partial'^U_k} & H_{k+1}(C_c^{\#}(\overline{U},A)) \end{array}$$

commutes. Since the diagram

$$\begin{array}{ccc} Q_c^{\#}(X,W) & \xrightarrow{j^X} & C_c^{\#}(X,\partial U) \\ {\scriptstyle \iota^{\#}_{X\overline{U}}}\downarrow & & \downarrow{\scriptstyle \iota^{\#}_{X\overline{U}}} \\ Q_c^{\#}(\overline{U},U) & \xrightarrow[j^{\overline{U}}]{} & C_c^{\#}(\overline{U},\partial U) \end{array}$$

commutes, we also have

$$(\iota^{\#}_{X\overline{U}})_* j^X_* = j^{\overline{U}}_* (\iota^{\#}_{X\overline{U}})_*.$$

Finally, since the diagram

$$\begin{array}{ccc} Q_c^{\#}(X,W) & \xrightarrow{\iota^{\#}_{XW}} & C_c^{\#}(W) \\ {\scriptstyle \iota^{\#}_{X\overline{U}}}\downarrow & & \downarrow{\scriptstyle \iota^{\#}_{WU}} \\ Q_c^{\#}(\overline{U},U) & \xrightarrow[\iota^{\#}_{\overline{U}U}]{} & C_c^{\#}(U) \end{array}$$

commutes, we have that

$$\iota^*_{WU}(\iota^{\#}_{XW})_* = (\iota^{\#}_{\overline{U}U})_*(\iota^{\#}_{X\overline{U}})_*.$$

Combining these observations, we have

$$\begin{aligned} \iota^*_{WU}\partial^X_k &= \iota^*_{WU} \circ (\iota^{\#}_{XW})_* \circ j^X_* \circ \partial'_k \\ &= (\iota^{\#}_{\overline{U}U})_* \circ (\iota^{\#}_{X\overline{U}})_* \circ j^X_* \circ \partial'_k = (\iota^{\#}_{\overline{U}U})_* \circ j^{\overline{U}}_* \circ (\iota^{\#}_{X\overline{U}})_* \circ \partial'_k \\ &= (\iota^{\#}_{\overline{U}U})_* \circ j^U_* \circ \partial'^U_k = \partial^U_k. \end{aligned}$$

This completes the proof. □

1.10 Homotopy property

An important property of (any) cohomology theory is that homotopic maps (in suitable sense) induce the same homomorphism on the level of cohomology. For the compactly supported Alexander–Spanier cohomology this holds for properly homotopic maps.

Definition 1.10.1. Proper continuous maps $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are *properly homotopic* if there exists a proper map $F\colon X\times[0,1] \to Y$ which is a homotopy from f_0 to f_1, that is, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for each $x \in X$.

In this section, we denote $I = [0,1]$. Given spaces X and Y, a mapping $F\colon X \times I \to Y$, and $t \in I$, we also denote $X_t = X \times \{t\} \subset X \times I$ and $F_t\colon X \to Y$ the map $x \mapsto (x,t)$.

Remark 1.10.2. *Note that a mapping $F\colon X\times I \to Y$ is proper if and only if each map $F_t\colon X \to Y$ is proper.*

We define the proper homotopy equivalence as usual.

Definition 1.10.3. A proper continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there exits a proper map $g\colon Y \to X$ (called *homotopy inverse of f* so that $g\circ f$ and $f \circ g$ are properly homotopic to identities id_X and id_Y, respectively.

The homotopy property of the compactly supported Alexander–Spanier cohomology now reads as follows.

Theorem 1.10.4. *Let X and Y be locally compact Hausdorff spaces and let $f_0\colon X \to Y$ and $f_1\colon X \to Y$ be properly homotopic maps. Then*

$$f_0^* = f_1^*\colon H_c^*(Y) \to H_c^*(X).$$

Corollary 1.10.5. *A homotopy equivalence $f\colon X \to Y$ induces an isomorphism $f^*\colon H_c^*(Y) \to H_c^*(X)$ in cohomology.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Then $f^*\circ g^* = (g\circ f)^* = \mathrm{id}_X^* = \mathrm{id}$ and $g^*\circ f^* = (f\circ g)^* = \mathrm{id}_Y^* = \mathrm{id}$. □

We begin the proof of Theorem 1.10.4 with an observation. For $i = 0,1$, let $h_i\colon X \to X\times I$ be the inclusion $x \mapsto (x,i)$. Let now $F\colon X\times I \to X$ is a proper homotopy from f_0 to f_1. Then $F\circ h_i = f_i$ and

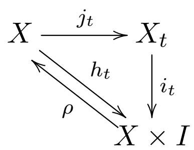
$$f_i^* = (F\circ h_i)^* = h_i^*\circ F^*$$

for $i = 0,1$. Thus it suffices to prove the following proposition.

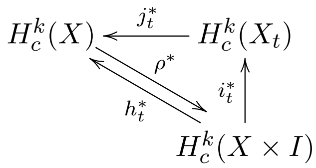
Proposition 1.10.6. *For $i = 0, 1$, let $h_i \colon X \to X \times I$ be the inclusion $x \mapsto (x, i)$. Then*
$$h_0^* = h_1^* \colon H_c^*(X \times I) \to H_c^*(X).$$

We begin by introducing some notation. For each $t \in I$, let
$$h_t \colon X \to X \times I, \quad x \mapsto (x, t),$$
and let $j_t \colon X \to X_t$ and $i_t \colon X_t \to X \times I$ be the homeomorphism $x \mapsto (x, t)$ and inclusion, respectively. So, formally, $h_t = i_t \circ j_t$ for each $t \in I$.

Let also
$$\rho \colon X \times I \to X, \quad (x, t) \mapsto x.$$
Thus we have the diagram



of maps, where j_t is a homeomorphism. Since all the maps in the diagram are proper, we have also a commutative diagram



in cohomology, where j_t^* is an isomorphism. Note also that
$$h_t^* \circ \rho^* = (\rho \circ h_t)^* = \mathrm{id}_X^* = \mathrm{id}$$
for each $t \in I$.

Homomorphisms ρ^* and h_t^* are homologically ortogonal in the following sense.[5]

Lemma 1.10.7. *For each $t \in I$,*
$$H_c^k(X \times I) = \operatorname{im} \rho^* \oplus \ker h_t^*.$$

Proof. We show first that $\operatorname{im} \rho^* \cap \ker h_t^* = \{0\}$. Let $c \in \operatorname{im} \rho^* \cap \ker h_t^*$. Then there exists $c' \in H_c^k(X)$ for which $\rho^*(c') = c$. Since $h_t^* \circ \rho^* = \mathrm{id}$, we have that
$$c' = h_t^*(\rho^*(c')) = h_t^*(c) = 0.$$

[5]Recall that, if U and V are subgroups of W, then $W = U \oplus V$ if $U \cap V = \{0\}$ and $U + V = W$.

Thus $c = \rho(c') = 0$ and $\operatorname{im} \rho^* \cap \ker h_t^* = \{0\}$.

To show that $H_c^k(X \times I) = \operatorname{im} \rho^* + \ker h_t^*$, let $c \in H_c^k(X \times I)$. Then

$$h_t^*(c - \rho^*(h_t^*(c))) = h_t^*(c) - (h_t^* \circ \rho^*)(h_t^*(c)) = h_t^*(c) - h_t^*(c) = 0.$$

Thus

$$c = \rho^*(h_t^*(c)) + (c - \rho^*(h_t^*(c))) \in \operatorname{im} \rho^* + \ker h_t^*.$$

This completes the proof. □

The main idea of the proof of Proposition 1.10.6 is that, for each $c \in H_c^k(X \times I)$, the map $I \to H_c^k(X)$, $t \mapsto h_t^*(c)$, is locally constant, and hence constant, since I is connected. This is immediately true for classes in the image of ρ^*. Indeed, we have the following lemma.

Lemma 1.10.8. *Let* $c \in \operatorname{im} \left(\rho^* \colon H_c^k(X) \to H_c^k(X \times I)\right)$. *Then*

$$h_t^* c = h_0^* c$$

for each $t \in I$.

Proof. Let $c' \in H_c^k(X)$ be a cohomology class and $c = \rho^*(c') \in H_c^k(X \times I)$. Since $h_t^* \circ \rho^* = \mathrm{id} = h_0^* \circ \rho^*$, we have

$$h_t^*(c) = h_t^*(\rho^*(c')) = c' = h_0^*(\rho^*(c')) = h_0^*(c).$$

The claim follows. □

We are now ready for the proof of Proposition 1.10.6.

Proof of Proposition 1.10.6. Let $c \in H_c^k(X \times I)$. We show that, for each t_0 has a neighborhood J in I satisfying $h_t^* c = h_{t_0}^* c$ for each $t \in J$. Since I is connected, a standard covering argument then implies the claim.

Let $t_0 \in I$. By Lemma 1.10.7, there exists $a \in \operatorname{im} \rho^*$ and $b \in \ker h_{t_0}^*$ for which $c = a + b$. Since $h_t^* a = h_0^* a$ for each $t \in I$ by Lemma 1.10.8 it suffices to find an interval $J \subset I$ containing t_0 in its interior for which we have $h_t^* b = h_{t_0}^* b = 0$ for all $t \in J$. Further, since $h_t^* = j_t^* \circ i_t^*$ and j_t^* is an isomorphism, it suffices to find an interval $J \subset I$, containing t_0 in its interior, for which $i_t^* b = i_{t_0}^* b = 0$ for each $t \in J$.

Consider the exact sequence of the pair $(X \times I, X_{t_0})$

$$\cdots \longrightarrow H_c^k((X \times I) \setminus X_{t_0}) \xrightarrow{\tau_{t_0}} H_c^k(X \times I) \xrightarrow{i_{t_0}^*} H_c^k(X_{t_0}) \xrightarrow{\partial_k} \cdots$$

By assumption, $b \in \ker i_{t_0}^*$. Thus there exists $b_1 \in H_c^k((X \times I) \setminus X_{t_0})$ for which $\tau_{t_0}(b_1) = b$. Since b_1 is a cohomology class in complactly supported cohomology there exists, by Lemma 1.6.1, a pre-compact open subset $U \subset$

$(X \times I) \setminus X_{t_0}$ for which b_1 is in the image of the push-forward $\tau_{(X\times I)U}$. Let $b_2 \in H^k_c(U)$ for which $b_1 = \tau_{((X\times I)\setminus X_{t_0})U}(b_2)$.

Since $\overline{U}$ is compact and $X_{t_0} \cap \overline{U} = \emptyset$, there exists a closed interval $J \subset I$ containing t_0 in its interior for which $(X \times J) \cap \overline{U} = \emptyset$. Indeed, let $p\colon X \times I \to I$ to be the projection $(x,t) \mapsto t$. Then $p(\overline{U})$ is compact and $t_0 \notin p(\overline{U})$. Thus there exists a closed interval J which contains t_0 in its interior and does not meet $p(\overline{U})$.

Let $\tau' = \tau_{(X\times(I\setminus J))U}$ and $b_3 = \tau_U(b_2) \in H^k_c(X \times (I \setminus J))$. By the naturality of the exact sequence of a pair (Theorem 1.9.3), we have the commuting diagram

$$\begin{array}{ccccccccc}
 & & H^k_c(U) & & & & & & \\
 & & \big\downarrow{\scriptstyle \tau'} & & & & & & \\
\cdots & \longrightarrow & H^k_c((X\times(I\setminus J)) & \xrightarrow{\tau_J} & H^k_c(X\times I) & \xrightarrow{i^*_J} & H^k_c(X\times J) & \xrightarrow{\partial_k} & \cdots \\
 & & \big\downarrow{\scriptstyle \tau} & & \big\|{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \kappa^*_{t_0}} & & \\
\cdots & \longrightarrow & H^k_c((X\times I)\setminus X_{t_0}) & \xrightarrow{\tau_{t_0}} & H^k_c(X\times I) & \xrightarrow{i^*_{t_0}} & H^k_c(X_{t_0}) & \xrightarrow{\partial_k} & \cdots
\end{array}$$

where rows are exact sequences of pairs $(X \times I, X \times J)$ and $(X \times I, X_{t_0})$, homomorphisms τ, τ_{t_0} and τ_J are the corresponding push-forward homomorphisms, and $i_J\colon X \times J \hookrightarrow X \times I$ and $\kappa_{t_0}\colon X_{t_0} \hookrightarrow X \times J$ inclusions.

By the composition laws and commutativity of the diagram, we have

$$\tau_J(b_3) \;=\; (\tau_{t_0} \circ \tau)(b_3) = (\tau_{t_0} \circ \tau \circ \tau')(b_2) = \tau_{(X\times I)U}(b_2) = b.$$

Let $t \in J$. We have the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^k_c((X\times(I\setminus J)) & \xrightarrow{\tau_J} & H^k_c(X\times I) & \xrightarrow{i^*_J} & H^k_c(X\times J) & \xrightarrow{\partial_k} & \cdots \\
 & & \big\downarrow{\scriptstyle \tau} & & \big\|{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \kappa^*_{t}} & & \\
\cdots & \longrightarrow & H^k_c((X\times I)\setminus X_{t}) & \xrightarrow{\tau_{t}} & H^k_c(X\times I) & \xrightarrow{i^*_{t}} & H^k_c(X_{t}) & \xrightarrow{\partial_k} & \cdots
\end{array}$$

where the rows are now exact sequences of pairs $(X \times I, X \times J)$ and $(X \times I, X_t)$, homomorphisms τ, τ_t and τ_J are the corresponding push-forward homomorphisms, and $i_J\colon X \times J \hookrightarrow X \times I$ and $\kappa_t\colon X_t \hookrightarrow X \times J$ inclusions.

Then

$$i^*_t(b) = i^*_t(\tau_J(b_3)) = (\kappa^*_t \circ i^*_J)(\tau_J(b_3)) = 0.$$

Thus $b \in \ker i^*_t$ for each $t \in J$. This concludes the proof. □

1.11 Mayer–Vietoris sequence

The Mayer–Vietoris sequence enables us to calculate the cohomology of a union $U \cup V$ from cohomologies of the (open) subsets U and V of X. The statement reads as follows. The exact sequence is called the *Mayer–Vietoris sequence.*

Theorem 1.11.1. *Let U and V be open sets in X for which $X = U \cup V$, and let*

$$\varphi\colon H^k_c(U \cap V) \to H^k_c(U) \oplus H^k_c(V), \quad c \mapsto (\tau_{U(U\cap V)}c, \tau_{V(U\cap V)}c)$$

and

$$\psi\colon H^k_c(U) \oplus H^k_c(V) \to H^k_c(X), \quad (a,b) \mapsto \tau_{XU}a - \tau_{XV}b.$$

Then, for each k, there exists a homomorphism $\Delta\colon H^k_c(X) \to H^{k+1}_c(U \cap V)$ for which the sequence

$$\xrightarrow{\Delta} H^k_c(U \cap V) \xrightarrow{\varphi} H^k_c(U) \oplus H^k_c(V) \xrightarrow{\psi} H^k_c(X) \xrightarrow{\Delta} H^{k+1}_c(U \cap V) \longrightarrow$$

is exact.

In the proof, we use the partition of unity.

Fact 1.11.2. *Let $\mathscr{U}$ be a finite open cover of a locally compact Hausdorff space X. Then there exists a partition of unity $\{\varphi_i\}_{i\in\mathcal{I}}$ with respect to $\mathscr{U}$, that is,*

1. *for each $i \in \mathcal{I}$, there exists $U_i \in \mathscr{U}$ for which $\mathrm{spt}(\varphi_i) \subset U_i$,*
2. *each $x \in X$ has a neighborhood $W_x \subset X$ for which*

$$\#\{i \in \mathcal{I}\colon \varphi_i|_{W_x} \neq 0\} < \infty, \quad \text{and}$$

3. *$\sum_{i\in\mathcal{I}} \varphi_i = 1$.*

The Mayer-Vietoris sequence stems from a short exact sequence for chain complexes.

Lemma 1.11.3. *Let U and V be open sets in X for which $X = U \cup V$, and denote $W = U \cap V$. Let also*

$$I\colon C^k_c(W) \to C^k_c(U) \oplus C^k_c(V), \quad c \mapsto ((\iota_{UW})_\#(c), (\iota_{VW})_\#(c))$$

and

$$J\colon C^k_c(U) \oplus C^k_c(V) \to C^k_c(X), \quad (a,b) \mapsto (\iota_{XU})_\#(a) - (\iota_{XV})_\#(b)$$

be homomorphisms. Then the sequence

$$0 \longrightarrow C^k_c(W) \xrightarrow{I} C^k_c(U) \oplus C^k_c(V) \xrightarrow{J} C^k_c(X) \longrightarrow 0$$

is exact.

Proof. Since $(\iota_{UV})_\#$ and $(\iota_{VW})_\#$ are injective, the homomorphism I is injective. Thus it suffices to show that the sequence is exact at $C^k_c(U)\oplus C^k_c(V)$ and that J is surjective.

By the composition law of the push-forward, we have

$$J \circ I = (\iota_{XU})_\# \circ (\iota_{UW})_\# - (\iota_{XV})_\# \circ (\iota_{VW})_\# = (\iota_{XW})_\# - (\iota_{XW})_\# = 0.$$

Thus $\operatorname{im} I \subset \ker J$.

Suppose $(a,b) \in \ker J$. Then $(\iota_{XU})_\#(a) = (\iota_{XV})_\#(b)$. Since $\operatorname{spt}((\iota_{XU})_\#(a)) \subset U$ and $\operatorname{spt}((\iota_{XV})_\#(b)) \subset V$, we conclude that $\operatorname{spt}((\iota_{XU})_\#(a)) = \operatorname{spt}((\iota_{XV})_\#(b)) \subset U \cap V = W$. Thus $a \in Q^k_c(U,W)$ and $b \in Q^k_c(V,W)$. Let $c_a \in C^k_c(W)$ and $c_b \in C^k_c(W)$ be such that $(\iota_{UW})_\#(c_a) = a$ and $(\iota_{VW})_\#(c_b) = b$. Since

$$(\iota_{XW})_\#(c_a) = (\iota_{XU})_\#((\iota_{UW})_\#(c_a)) = (\iota_{XV})_\#((\iota_{VW})_\#(c_b)) = (\iota_{XW})_\#(c_b),$$

and $(\iota_{XW})_\#$ is injecive, we conclude that $c_a = c_b$. Thus

$$I(c_a) = ((\iota_{UW})_\#(c_a), (\iota_{VW})_\#(c_a)) = (a, (\iota_{VW})_\#(c_b)) = (a,b).$$

Hence $\ker J \subset \operatorname{im} I$.

To show the surjectivity of J, let $c \in C^k_c(X)$. We also fix $\phi \in \Phi^k_c(X)$ so that $c = [\phi]$. We construct $\phi_U \in \Phi^k_c(X,U)$ and $\phi_V \in \Phi^k_c(X,V)$ for which

$$[\phi_U] - [\phi_V] = [\phi] = c.$$

Then, for $a = (\iota_{XU})^\#([\phi_U]) \in C^k_c(U)$ and $b = (\iota_{XV})^\#([\phi_V]) \in C^k_c(V)$, we have

$$J(a,b) = (\iota_{XU})_\#(a) - (\iota_{XV})_\#(b) = [\phi_U] - [\phi_V] = c.$$

Let $\{\lambda_U, \lambda_V\}$ be a partition of unity on X with respect to the cover $\{U,V\}$ of X satisfying $\operatorname{spt}(\lambda_U) \subset U$ and $\operatorname{spt}(\lambda_V) \subset V$. Let $\pi \colon X^{k+1} \to X$ be the projection $(x_1,\ldots,x_{k+1}) \mapsto x_1$ and let $\phi_U = (\lambda_U \circ \pi)\phi$ and $\phi_V = -(\lambda_V \circ \pi)\phi$ be k-functions in $\Phi^k(X)$. Since

$$\operatorname{spt}(\phi_U) \subset \operatorname{spt}(\lambda_U) \cap \operatorname{spt}(\phi) \quad \text{and} \quad \operatorname{spt}(\phi_V) \subset \operatorname{spt}(\lambda_V) \cap \operatorname{spt}(\phi).$$

we have $\phi_U \in \Phi^k_c(X,U)$ and $\phi_V \in \Phi^k_c(X,V)$.

We show that $[\phi] = [\phi_U] - [\phi_V]$. Let $x \in X$. Then there exists a neigborhood B of x for which $\lambda_U|_{B^{k+1}} + \lambda_V|_{B^{k+1}} = 1$. Thus

$$\begin{aligned} \phi_U|_{B^{k+1}} - \phi_V|_{B^{k+1}} &= ((\lambda_U \circ \pi)\phi))|_{B^{k+1}} - (-(\lambda_V \circ \pi)\phi)|_{B^{k+1}} \\ &= ((\lambda_U + \lambda_V) \circ \pi)\,|_{B^{k+1}}\phi|_{B^{k+1}} = \phi_{B^{k+1}}. \end{aligned}$$

Thus $x \in \operatorname{null}(\phi - (\phi_U - \phi_V))$. Hence $[\phi] = [\phi_U] - [\phi_V]$. This concludes the proof. $\square$

Proof of Theorem 1.11.1. By Lemma 1.11.3, we have a long exact sequence

$$\cdots \xrightarrow{\partial} H_c^k(U \cap V) \xrightarrow{I_*} H_c^k(C_c^*(U) \oplus C_c^*(V)) \xrightarrow{J_*} H_c^k(X) \xrightarrow{\partial} H_c^{k+1}(U \cap V) \longrightarrow \cdots$$

in homology. Let now $\theta\colon H_c(C_c^*(U) \oplus C_c^*(V)) \to H_c^k(U) \oplus H_c^k(V)$ be the natural isomorphism. Then the diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial} & H_c^k(U \cap V) & \xrightarrow{I_*} & H_c^k(C_c^*(U) \oplus C_c^*(V)) & \xrightarrow{J_*} & H_c^k(X) & \xrightarrow{\partial} & H_c^{k+1}(U \cap V) & \longrightarrow \cdots \\
& \| & & \theta \downarrow \cong & & \| & & \| & \\
\cdots \xrightarrow{\partial} & H_c^k(U \cap V) & \xrightarrow{\varphi} & H_c^k(U) \oplus H_c^k(V) & \xrightarrow{\psi} & H_c^k(X) & \xrightarrow{\partial} & H_c^{k+1}(U \cap V) & \longrightarrow \cdots
\end{array}$$

commutes. □

Chapter 2

Orientation of domains Euclidean spaces

2.1 Goals

The goal of this chapter is to provide the existence of the so-called orientataion classes. More formally, the goal of this section is to establish the following theorem.

Theorem 2.1.1. *Let U be a connected open set in $\mathbb{R}^n$. Then $H^n_c(U) \cong \mathbb{Z}$ and the push-forward $\tau_{\mathbb{R}^n U} \colon H^n_c(U) \to H^n_c(\mathbb{R}^n)$ is an isomorphism.*

Corollary 2.1.2. *Let $V \subset U \subset \mathbb{R}^n$ be open sets. Then $\tau_{UV} \colon H^n_c(V) \to H^n_c(U)$ is an isomorphism.*

Proof. Since $\tau_{\mathbb{R}^n V} = \tau_{\mathbb{R}^n U} \circ \tau_{UV}$, the claim follows. □

Having these results at our disposal, we may define an orientation class of a domain.

Definition 2.1.3. Let U be a domain[1] either in $\mathbb{R}^n$ or in $\mathbb{S}^n$. A choice of generator $c_U \in H^n_c(U) \cong \mathbb{Z}$ is called an *orientation (class) of U*. Domains $V \subset U$ are *consistently oriented* if $\tau_{UV}(c_V) = c_U$.

As one step of the proof of Theorem 2.1.1, we show that all higher cohomology groups $H^k_c(U)$ for $k > n$ of an open set U in $\mathbb{R}^n$ vanish.

Theorem 2.1.4. *Let $U \subset \mathbb{R}^n$ be an open set. Then $H^k_c(U) = 0$ for $k > n$. Similarly, if $V \subset \mathbb{S}^n$ is an open set, then $H^k_c(V) = 0$ for $k > n$.*

As an important corollary of this vanishing result, we have a corresponding result for closed sets. For its importance, we regard it also as a theorem.[2]

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[1] An open and connected subset is called a *domain*.

[2] This result is **NOT** true for the singular cohomology, see Barratt and Milnor *Proc. Amer. Math. Soc.* 13 (1963), 293–297.

Theorem 2.1.5. *Let $n \geq 1$ and $A \subset \mathbb{S}^n$ be a closed set which is not the whole sphere. Then $H_c^k(A) = 0$ for $k \geq n$.*

The corresponding (general) results hold also for $n$manifolds and we disucss them in a separate section in the end of this chapter.

Theorem 2.1.1 is proved in three steps. First we calculate cohomology groups $H_c^*(\mathbb{R}^n)$ and $H_c^*(\mathbb{S}^n)$ of $\mathbb{R}^n$ and $\mathbb{S}^n$, respectively. Then we show in Theorem 2.1.4 that higher cohomology groups ($k > n$) of open sets in $\mathbb{R}^n$ and $\mathbb{S}^n$ are trivial and discuss its applications to closed sets in $\mathbb{R}^n$ and $\mathbb{S}^n$. Then we are ready to prove Theorem 2.1.1.

2.2 Cohomology groups of Euclidean spaces

In this section, we calculate the cohomology (rings) of the Euclidean n-space $\mathbb{R}^n$ and the n-sphere $\mathbb{S}^n$; recall that $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \colon |x| = 1\}$. The results (as expected) read as follows. We restrict ourselves to the case $n \geq 1$. For $n = 0$, we have that $\mathbb{R}^0$ is a point and $\mathbb{S}^0$ is a disjoint union of two points. Thus the following results hold with the exception that $H_c^0(\mathbb{S}^0) \cong \mathbb{Z}^2$.

Theorem 2.2.1. *Let $n \geq 1$. Then*

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & \text{for } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 2.2.2. *Let $n \geq 1$. Then*

$$H_c^k(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z}, & \text{for } k = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

The underlying fact behind the proof of Theorem 2.2.1 is the triviality of the cohomology of an n-cell. Interestingly, this fact is a mere observation and we state it as such.

Observation 2.2.3. *Let $x_0 \in \bar{B}^n$, then the inclusion $\iota \colon \{x_0\} \to \bar{B}^n$ is a proper homotopy equivalence. In particular, $\iota^* \colon H_c^*(\bar{B}^n) \to H_c^*(\{x_0\})$ is an isomorphism and we have*

$$H_c^k(\bar{B}^n) \cong H_c^k(\{x_0\}) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Finally, one more lemma before the proof of Theorem 2.2.1.

Lemma 2.2.4. *Let $n \geq 1$ and $x_0 \in \bar{B}^n$. Then the push-forward homomorphism $\tau \colon H_c^k(\bar{B}^n \setminus \{x_0\}) \to H_c^k(\bar{B}^n)$ is an isomorphism for each $k > 0$. In particular, $H_c^k(\bar{B}^n \setminus \{x_0\}) = 0$ for all $k \geq 0$.*

Proof. Since $D = \bar{B}^n \setminus \{x_0\}$ is not compact, $H_c^0(\bar{B}^n \setminus \{x_0\}) = 0$. By the exact sequence of the pair $(\bar{B}^n, \{x_0\})$, we have

$$\cdots \longrightarrow H_c^{k-1}(\{x_0\}) \xrightarrow{\partial_{k-1}} H_c^k(D) \xrightarrow{\tau} H_c^k(\bar{B}^n) \xrightarrow{\iota^*} H_c^k(\{x_0\}) \xrightarrow{\partial_k} \cdots$$

where $\tau = \tau_{\bar{B}^n D}$ is a push-forward and $\iota = \iota_{\bar{B}^n, \{x_0\}}$ an inclusion.

Since $H_c^\ell(\{x_0\}) = 0$ for $\ell > 0$, we have

$$0 \longrightarrow H_c^k(D) \xrightarrow{\tau} H_c^k(\bar{B}^n) \longrightarrow 0$$

for $k > 1$. Thus the claim holds for $k > 1$.

To prove the claim for $k = 1$, it suffices to show that $H_c^1(D) = 0$. We note first that $H_c^0(D) = 0$ and $H_c^1(\bar{B}^n) = 0$. Thus we have the exact sequence

$$0 \longrightarrow H_c^0(\bar{B}^n) \xrightarrow{\iota^*} H_c^0(\{x_0\}) \xrightarrow{\partial} H_c^1(D) \longrightarrow 0$$

where ι is an isomorphism by Observation 2.2.3. Thus $\ker \partial = H_c^0(\{x_0\})$ and $\partial = 0$. Hence $H_c^1(D) = \operatorname{im} \partial = 0$.

The second claim now follows immediately. □

Having Lemma 2.2.4 at our disposal, we are ready for the proof of Theorem 2.2.1. For the proof, we record a fact.

Fact 2.2.5. *For $n \geq 1$, there exists a homeomorphism $\pi \colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ (called* stereogrphic projection*) for which $\pi(S_\pm) = \mathbb{R}^n_\pm$, where*

- $\mathbb{R}^n_\pm = \{(x_1, \ldots, x_n) \colon \mathbb{R}^n \colon x_n = \pm |x_n|\}$, *and*
- $S_\pm = \{(y_1, \ldots, y_{n+1}) \in \mathbb{S}^n \colon y_n = \pm |y_n|\}$.

In particular, $\mathbb{S}^n$ is a one-point compactification of $\mathbb{R}^n$.

Proof of Theorem 2.2.1. Since $\mathbb{R}^0$ is a point, the claim holds for $n = 0$. Let $n \geq 1$ and suppose that the claim holds for $n - 1$.

Let $\mathbb{R}^n_\pm = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n = \pm |x_n|\}$ be as in Fact 2.2.5. Then $\mathbb{R}^n_+ \cap \mathbb{R}^n_- = \mathbb{R}^{n-1} \times \{0\} \approx \mathbb{R}^{n-1}$. Let also $D_\pm = \mathbb{R}^n_\pm \setminus (\mathbb{R}^n_+ \cap \mathbb{R}^n_-)$. Then $D_\pm \approx \mathbb{R}^n$. Moreover, $\mathbb{R}^n_\pm \approx \bar{B}^n \setminus \{x_0\}$, where $x_0 \in \partial \bar{B}^n$.

Let $A = \mathbb{R}^{n-1} \times \{0\}$ and $\iota_\pm \colon A \hookrightarrow \mathbb{R}^n_\pm$ an inclusion. By the long exact sequence of a pair $(\mathbb{R}^n_\pm, A)$, we have the exact sequence

$$H_c^k(\mathbb{R}^n_\pm) \xrightarrow{\iota^*} H_c^k(A) \xrightarrow{\partial_k} H_c^{k+1}(D_\pm) \xrightarrow{\tau} H_c^{k+1}(\mathbb{R}^n_\pm)$$

By Lemma 2.2.4,

$$H_c^k(\mathbb{R}^n_\pm) \cong H_c^k(\bar{B}^n \setminus \{x_0\}) = 0$$

for $k \in \mathbb{Z}$. Thus, for $k \in \mathbb{Z}$, we have

$$0 \longrightarrow H_c^k(A) \xrightarrow{\partial_k} H_c^{k+1}(D_\pm) \longrightarrow 0$$

Thus $\partial_k \colon H_c^k(A) \to H_c^{k+1}(D_\pm)$ is an isomorphism for all $k \in \mathbb{Z}$. Thus

$$H_c^k(\mathbb{R}^n) \cong H_c^k(A) \cong H_c^{k+1}(D_\pm) \cong H_c^{k+1}(\mathbb{R}^{n+1})$$

for all $k \in \mathbb{Z}$. □

Proof of Corollary 2.2.2. We know that $H_c^0(\mathbb{S}^n) \cong \mathbb{Z}$ by connectedness and compactness. We also know that $\mathbb{R}^n \approx \mathbb{S}^n \setminus \{e_{n+1}\}$. Thus it suffices to show that $H_c^k(\mathbb{S}^n \setminus \{e_{n+1}\}) \cong H_c^k(\mathbb{S}^n)$ for $k > 0$.

We have the exact sequence

$$H_c^{k-1}(\{e_{n+1}\}) \xrightarrow{\partial_{k-1}} H_c^k(\mathbb{S}^n \setminus \{e_{n+1}\}) \xrightarrow{\tau} H_c^k(\mathbb{S}^n) \xrightarrow{\iota^*} H_c^k(\{e_{n+1}\})$$

of the pair $(\mathbb{S}^n, \{e_{n+1}\})$, where $\iota \colon \{e_{n+1}\} \hookrightarrow S^n$ is an inclusion.

For $k > 1$, τ is clearly an isomorphism. For $k = 1$, $\iota^* \colon H_c^0(\mathbb{S}^n) \to H_c^0(\{e_{n+1}\})$ is an isomorphism by Lemma 1.4.12. Thus $\partial_0 = 0$. Since $H_c^1(\{e_{n+1}\}) = 0$, we have

$$0 \longrightarrow H_c^1(\mathbb{S}^n \setminus \{e_{n+1}\}) \xrightarrow{\tau} H_c^1(\mathbb{S}^n) \longrightarrow 0$$

and $\tau \colon H_c^1(\mathbb{S}^n \setminus \{e_{n+1}\}) \to H_c^1(\mathbb{S}^n)$ is an isomorphism. □

2.3 Vanishing above top dimension: open sets

We prove now the vanishing of higher cohomology in the case of open sets of $\mathbb{R}^n$. The argument reduces to the case of cubical open sets. An open set $Q \subset \mathbb{R}^n$ is a *dyadic cube* if there exists $v \in \mathbb{Z}^n$ and $k \geq 0$ for which $Q = 2^{-k}v + (0, 2^{-k})^n$. Let $\mathcal{D}(\mathbb{R}^n)$ be the set of all dyadic cubes in $\mathbb{R}^n$.

Definition 2.3.1. An open set $U \subset \mathbb{R}^n$ is *cubical* if there exists a finite subset $\mathcal{C} \subset \mathcal{D}(\mathbb{R}^n)$ so that elements in $\mathcal{C}$ are pair-wise disjoint (i.e. $Q \cap Q' = \emptyset$ for $Q \neq Q'$) and U is the interior of the set $\bigcup_{Q \in \mathcal{C}} \overline{Q}$. We call $\mathcal{C}$ an *dyadic partition of* U.

Theorem 2.1.4. Let $U \subset \mathbb{R}^n$ be an open set. Then $H_c^k(U) = 0$ for $k > n$. Similarly, if $V \subset \mathbb{S}^n$ is an open set, then $H_c^k(V) = 0$ for $k > n$.

Proof. The second claim follows immediately from the first and thus it suffices to show that $H_c^k(U) = 0$ for an open set $U \subset \mathbb{R}^n$ and $k > n$.

The proof is an induction by dimension. Suppose that $n = 1$. Since components of (non-empty) open sets in $\mathbb{R}$ are open intervals, and open intervals are homeomorphic to $\mathbb{R}$. By Theorem 2.2.1, the claim follows for $n = 1$.

Suppose now that, for $n > 1$, the claim holds for all open sets in $\mathbb{R}^{n-1}$. We prove the claim first for cubical sets in $\mathbb{R}^n$.

Since a dyadic cube is homeomorphic to $\mathbb{R}^n$, we conclude that the claim holds for each dyadic cube in $\mathbb{R}^n$. Suppose now, for $k \geq 1$, the claim holds in $\mathbb{R}^n$ for all open cubical sets which have a partition into k dyadic cubes. Let U be an open cubical set having a dyadic partition $\mathcal{D}$ with $k+1$ elements. Let now $Q \in \mathcal{D}$ be a dyadic cube having the smallest diameter. Set $\mathcal{D}' = \mathcal{D} \setminus \{Q\}$ and let U' be the interior of the set $\bigcup_{Q' \in \mathcal{D}'} \overline{Q'}$. Let $A = (\overline{Q} \cap U) \setminus Q$. Then A is closed in U and open in ∂Q.

We have, by the exact sequence of the pair (U, A), that the sequence

$$H^k_c(U \setminus A) \xrightarrow{\tau} H^k_c(U) \xrightarrow{\iota^*} H^k_c(A) \tag{2.3.1}$$

is exact, where $\iota \colon A \hookrightarrow U$ is an inclusion and τ is the push-forward.

Since $U \setminus A = U' \cup Q$ and $U' \cap Q = \emptyset$, we have, by the induction assumption and the fact $Q \approx \mathbb{R}^n$, that

$$H^k_c(U \setminus A) \cong H^k_c(U') \oplus H^k_c(Q) = 0$$

for $k > n$.

On the other hand, $A \subset \partial Q$ is open and $\partial Q \approx \mathbb{S}^{n-1}$. If $A = \partial Q$, we conclude that $H^k_c(A) \cong H^k_c(\mathbb{S}^{n-1}) = 0$ for $k \geq n$. If $A \neq \partial Q$, then A is homeomorphic to an open set in $\mathbb{R}^n$. Thus $H^k_c(A) = 0$ for $k \geq n$ by the induction assumption. Then, by (2.3.1), $H^k_c(U) = 0$ for $k > n$.

We complete now the induction step by considering an open set $U \subset \mathbb{R}^n$. Let $k > n$ and $c \in H^k_c(U)$. Then there exists an open set $V \subset U$ so that $\overline{V}$ is compact and $c \in \operatorname{im} \tau_{UV}$. Let now $\mathcal{C} \subset \mathcal{D}$ be a finite collection of pair-wise disjoint dyadic cubes for which $\overline{V}$ is contained in $W = \operatorname{int} \bigcup_{Q \in \mathcal{C}} \overline{Q}$. Then $c \in \operatorname{im} \tau_{UW} = \{0\}$. Thus $H^k_c(U) = 0$ for all $k > n$. This completes the induction step and the proof. □

2.4 Vanishing above top dimension: closed sets

We use now Theorem 2.1.4 to prove the vanishing of the higher cohomology of the closed sets. More precisely, we prove the following.

Theorem 2.1.5. Let $n \geq 1$ and $A \subset \mathbb{S}^n$ be a proper[3] closed subset. Then $H^k_c(A) = 0$ for $k \geq n$.

For the proof of Theorem 2.1.5, we record an observation.

[3] In the sense that $A \neq \mathbb{S}^n$

Observation 2.4.1. *Let $A \subset \mathbb{R}^n$ be a compact set. Then the inclusion map $\iota\colon A \hookrightarrow \mathbb{R}^n$ is properly homotopic to a constant map. Further, let $A \subset \mathbb{S}^n$ be a closed set which is not the whole sphere and $x_0 \in \mathbb{S}^n \setminus A$. Then $\iota\colon A \hookrightarrow \mathbb{S}^n \setminus \{x_0\}$ is properly homotopic to a constant map $A \to \mathbb{S}^n \setminus \{x_0\}$.*

Proof of Theorem 2.1.5. Let $j\colon A \hookrightarrow \mathbb{S}^n$ be an inclusion. By the exact sequence of the pair $(\mathbb{S}^n, A)$, we have

$$H_c^k(\mathbb{S}^n) \xrightarrow{j^*} H_c^k(A) \xrightarrow{\partial_k} H_c^{k+1}(\mathbb{S}^n \setminus A) \xrightarrow{\tau} H_c^{k+1}(\mathbb{S}^n),$$

where $H_c^k(\mathbb{S}^n) = H_c^{k+1}(\mathbb{S}^n) = 0$ for $k > n$. Thus, for $k > n$, $\partial_k\colon H_c^k(A) \to H_c^{k+1}(\mathbb{S}^n \setminus A)$ is an isomorphism and $H_c^k(A) = 0$ by Theorem 2.1.4.

For $k = n$, we argue as follows. Let $x_0 \in \mathbb{S}^n \setminus A$ and $i\colon \mathbb{S}^n \setminus \{x_0\} \to \mathbb{S}^n$ an inclusion. Then

$$\begin{array}{ccc} A & \xrightarrow{\quad j \quad} & \mathbb{S}^n \\ & {\scriptstyle \iota}\searrow \quad \nearrow{\scriptstyle i} & \\ & \mathbb{S}^n \setminus \{x_0\} & \end{array}$$

where $\iota\colon A \hookrightarrow \mathbb{S}^n \setminus \{x_0\}$ is an inclusion. Since ι is properly homotopic to a constant map by Observation 2.4.1, the pull-back $\iota^*\colon H_c^n(\mathbb{S}^n \setminus \{x_0\}) \to H_c^n(A)$ is the zero map. Hence $j^* = \iota^* \circ i^* = 0$, and $\partial_n\colon H_c^n(A) \to H_c^{n+1}(\mathbb{S}^n \setminus A)$ is an isomorphism. Thus $H_c^n(A) = 0$ by Theorem 2.1.4. □

Corollary 2.4.2. *Let $A \subset \mathbb{R}^n$ be a closed subset. Then $H_c^k(A) = 0$ for $k \geq n$.*

Proof. Let $\pi\colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ be the stereographic projection and $A' = \pi^{-1}A$. If A' is compact, then $H_c^k(A) \cong H_c^k(A') = 0$ for $k \geq n$ by Theorem 2.1.5. Suppose A' is not compact. Then $\overline{A'} = A' \cup \{e_{n+1}\}$ is compact and $H_c^k(\overline{A'}) = 0$ for $k \geq n$. For $n \geq 2$, it follows now from the exact sequence

$$H_c^{k-1}(\{e_{n+1}\}) \xrightarrow{\partial_k} H_c^k(A') \xrightarrow{\tau} H_c^k(\overline{A'}) \xrightarrow{\iota^*} H_c^k(\{e_{n+1}\})$$

of the pair $(\overline{A'}, \{e_{n+1}\})$ that $\tau\colon H_c^n(A') \to H_c^n(\overline{A'})$ is an isomorphism.

For $n = 1$, we use again Lemma 1.4.12 to observe that $\iota^*\colon H_c^0(\overline{A'}) \to H_c^0(\{e_{n+1}\})$ is an isomorphism. Thus $\tau\colon H_c^1(A') \to H_c^1(\overline{A'})$ is an isomorphism. Thus, for $n \geq 1$, the claim now follows from Theorem 2.1.5. □

2.5 Push-forward in the top dimension

In this section we finish the proof of Theorem 2.1.1.

Theorem 2.1.1. Let U be a connected open set in $\mathbb{R}^n$. Then $H_c^n(U) \cong \mathbb{Z}$ and the push-forward $\tau_{\mathbb{R}^n U}\colon H_c^n(U) \to H_c^n(\mathbb{R}^n)$ is an isomorphism.

The corresponding result for the n-sphere is an immediate corollary.

Corollary 2.5.1. *Let $U \subset \mathbb{S}^n$ be an open set. Then $\tau_{\mathbb{S}^n U} \colon H^n_c(U) \to H^n_c(\mathbb{S}^n)$ is an isomorphism.*

Proof. We may assume that $U \neq \mathbb{S}^n$. Let $x_0 \in \mathbb{S}^n \setminus U$. By the long exact sequence for the pair $(\mathbb{S}^n, \{x_0\})$, the push-forward $\tau \colon H^n_c(\mathbb{S}^n \setminus \{x_0\}) \to H^n_c(\mathbb{S}^n)$ is an isomorphism. Since $\mathbb{S}^n \setminus \{x_0\}$ is homeomorphic to $\mathbb{R}^n$, we conclude that the push-forward $\tau_{(\mathbb{S}^n \setminus \{x_0\} U} \colon H^n_c(U) \to H^n_c(\mathbb{S}^n \setminus \{x_0\})$ is an isomorphism. The claim now follows. □

An important corollary of Theorem 2.1.1 is that closed subsets of domains in $\mathbb{R}^n$ do not carry higher cohomology. We record this corollary as follows.

Corollary 2.5.2. *Let $U \subset \mathbb{S}^n$ be a domain and $A \subset U$ a proper closed subset in U. Then $H^k_c(A) = 0$ for $k \geq n$.*

Proof. We may assume that U is a proper subset of $\mathbb{S}^n$. Let $k \geq n$ and consider the exact sequence

$$H^k_c(U \setminus A) \xrightarrow{\tau} H^k_c(U) \xrightarrow{\iota^*} H^k_c(A) \xrightarrow{\partial} H^{k+1}_c(U \setminus A).$$

For $k > n$, $H^k_c(U) = 0$ and $H^{k+1}_c(U \setminus A) = 0$, since U and $U \setminus E$ are proper open subsets of $\mathbb{S}^n$. Thus $H^k_c(A) = 0$ by exactness.

Suppose now that $k = n$. Since $U \setminus A \neq \emptyset$ and U is connected, the push-forward $\tau \colon H^n_c(U \setminus A) \to H^n_c(U)$ is surjective. Thus ι^* is the zero map by exactness. Since $H^{n+1}_c(U \setminus A) = 0$, we conclude that $H^n_c(A) = 0$. □

Heuristically, Theorem 2.1.1 stems from the observation that each Euclidean ball contained in the domain U carries the nth compactly supported cohomology of U. More precisely, we show the following lemma and proposition.

Lemma 2.5.3. *Let B be a Euclidean ball in $\mathbb{R}^n$. Then the push-forward $\tau_{\mathbb{R}^n B} \colon H^n_c(B) \to H^n_c(\mathbb{R}^n)$ is an isomorphism.*

Proposition 2.5.4. *Let U be a domain in $\mathbb{R}^n$ and $B \subset U$ a Euclidean ball compactly contained in U. Then $\tau_{UB} \colon H^n_c(B) \to H^n_c(U)$ is surjective.*

Having these results we easily finish the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Let B be a Euclidean ball compactly contained in U. By Lemma 2.5.3, the push-forward $\tau_{UB} \colon H^n_c(B) \to H^n_c(\mathbb{R}^n)$ is an isomorphism. Since $\tau_{\mathbb{R}^n B} = \tau_{\mathbb{R}^n U} \tau_{UB}$, the push-forward $\tau_{\mathbb{R}^n U} \colon H^n_c(U) \to H^n_c(\mathbb{R}^n)$ is surjective. Since $\tau_{QB} \colon H^n_c(B) \to H^n_c(U)$ is surjective by Proposition 2.5.4 and $\tau_{\mathbb{R}^n B}$ has trivial kernel, we have that $\tau_{\mathbb{R}^n U} \colon H^n_c(U) \to H^n_c(\mathbb{R}^n)$ is injective. Thus $\tau_{\mathbb{R}^n U}$ is an isomorphism and $H^n_c(U)$ is infinite cyclic. □

It remains to prove Lemma 2.5.3 and Proposition 2.5.4. We begin with a sharper version of Lemma 2.5.3.

Lemma 2.5.5. *Let B be a Euclidean ball compactly contained in the Euclidean unit ball B^n, that is, $\overline{B} \subset B^n$. Then the push-forward $\tau_{B^n B} \colon H_c^n(B) \to H_c^n(B^n)$ is an isomorphism. Furthermore, $\tau_{\mathbb{R}^n B} \colon H_c^n(B) \to H_c^n(\mathbb{R}^n)$ is an isomorphism.*

Proof. Let $A = \bar{B}^n \setminus B$. We observe first that the inclusion $\iota \colon S^{n-1} \to A$ is a proper homomotopy equivalence. Indeed, let x_0 be the center of B and $p \colon A \to S^{n-1}$ the radial projection from x_0 to S^{n-1}. Then $p \circ \iota = \mathrm{id}_{S^{n-1}}$ and $\iota \circ p$ is properly homotopic to id_A.

Since ι is a proper homotopy equivalence, we have that $\iota^* \colon H_c^n(A) \to H_c^n(S^{n-1})$ is an isomorphism. Thus, by the naturality of the exact sequence of pairs, the diagram

$$\begin{array}{ccccccccc}
H_c^{n-1}(\bar{B}^n) & \xrightarrow{\iota^*_{\bar{B}^n A}} & H_c^{n-1}(A) & \xrightarrow{\partial} & H_c^n(B) & \xrightarrow{\tau_{\bar{B}^n B}} & H_c^n(\bar{B}^n) & \xrightarrow{\iota^*_{\bar{B}^n A}} & H_c^n(A) \\
\| & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \tau} & & \| & & \downarrow{\scriptstyle \iota^*} \\
H_c^{n-1}(\bar{B}^n) & \xrightarrow{\iota^*_{\bar{B}^n S^{n-1}}} & H_c^{n-1}(S^{n-1}) & \xrightarrow{\partial} & H_c^n(B^n) & \xrightarrow{\tau_{\bar{B}^n B^n}} & H_c^n(\bar{B}^n) & \xrightarrow{\iota^*_{\bar{B}^n S^{n-1}}} & H_c^n(S^{n-1})
\end{array}$$

commutes. Since $H_c^n(\bar{B}^n) = H_c^n(A) = H_c^n(S^{n-1}) = 0$ and ι^* is an isomorphism, $\tau_{\bar{B}^n B}$ is an isomorphism by the 5-lemma.

The second statement follows by an analogous argument; consider $A = \mathbb{R}^n \setminus B$ and the set $A' = \mathbb{R}^n \setminus B^n$ in place of $\mathbb{S}^{n-1}$. □

For the proof of Proposition 2.5.4 we show that all Euclidean ball compactly contained in U carry the same cohomology. More precisely, we prove the following.

Lemma 2.5.6. *Let U be a domain in $\mathbb{R}^n$. Each push-forward $\tau_{UB} \colon H_c^n(B) \to H_c^n(U)$, where B is a Euclidean ball compactly contained in U, has the same image.*

Proof. Let B and B' be dyadic cubes compactly contained in U for which $B \cap B' \neq \emptyset$. Then there exists a Euclidean ball B_0, compactly contained in $B \cap B'$. Then, by Lemma 2.5.5, we have that

$$\operatorname{im} \tau_{UB} = \operatorname{im} \tau_{UB_0} = \operatorname{im} \tau_{UB'}.$$

Since U is connected, the claim follows. □

Lemma 2.5.7. *Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be finite collection of Euclidean balls in $\mathbb{R}^n$ having connected union $V = \bigcup \mathcal{B}$, and let B be a Euclidean ball compactly contained in V. Then $\tau_{VB} \colon H_c^n(B) \to H_c^n(V)$ is surjective.*

Proof. By Lemma 2.5.5 the claim holds for $k = 1$. Suppose now that the claim holds for all collections of k balls. Let $\mathcal{B}$ be a collection of $k+1$ balls having connected union V. Let also B be a Euclidean ball compactly contained in V. Let $B_1, \ldots, B_{k+1}$ be such a labeling of balls in $\mathcal{B}$ that collection $\mathcal{B}' = \{B_1, \ldots, B_k\}$ has connected union $V' = \bigcup \mathcal{B}'$. Let also B' be a ball compactly contained in $V' \cap B_{k+1}$. Then, by the induction assumption, $\tau_{V'B'} \colon H^n_c(B') \to H^n_c(V')$ is surjective. Note that, the push-forward $\tau_{B_{k+1},B'} \colon H^n_c(B_{k+1})$ is surjective by Lemma 2.5.5.

We show now that $\tau_{VV'} \colon H^n_c(V') \to H^n_c(V)$ and $\tau_{VB_{k+1}} \colon H^n_c(B_{k+1}) \to H^n_c(V)$ are surjective. Let $c \in H^n_c(V)$.

Since $H^{n+1}_c(V' \cap B_{k+1}) = 0$, we have, by the Mayer-Vietoris sequence, that the homomorphism $\psi \colon H^n_c(V') \oplus H^n_c(B_{k+1}) \to H^n_c(V)$, $(a,b) \mapsto \tau_{VV'}a - \tau_{VB_{k+1}}b$, is surjective. Thus there exists $a \in H^n_c(V')$ and $b \in H^n_c(B_{k+1})$ for which $\tau_{VV'}a - \tau_{VB_{k+1}}b = c$.

Since $\tau_{V'B'}$ and $\tau_{B_{k+1}B'}$ are surjective, there exists a' and b' in $H^n_c(B')$ for which $\tau_{V'B'}a' = a$ and $\tau_{B_{k+1}B'}b' = b$. Thus

$$\tau_{VV'}(a - \tau_{V'B'}b') = \tau_{VV'}a - \tau_{VB'}b' = \tau_{VV'}a - \tau_{VB_{k+1}}b = c$$

and

$$\tau_{VB_{k+1}}(\tau_{B_{k+1}B'}a' - b) = \tau_{VB'}a' - \tau_{VB_{k+1}}b = \tau_{VV'}a - \tau_{VB_{k+1}}b = c.$$

Thus $\tau_{VV'}$ and $\tau_{VB_{k+1}}$ are surjective.

We consider now two cases. Suppose first that $B \subset B_{k+1}$. Then the claim follows from the surjectivity of $\tau_{VB_{k+1}}$ and Lemma 2.5.5. Suppose now that $B \cap V' \neq \emptyset$. Then there exists a Euclidean ball B'' compactly contained in $V' \cap B$. By induction assumption, $\tau_{V'B''}$ is surjective. Thus $\tau_{VB''} = \tau_{VV'} \circ \tau_{V'B''}$ is surjective. Since $\tau_{VB''} = \tau_{VB} \circ \tau_{BB''}$, we conclude that τ_{VB} is surjective. This completess the proof. □

Proof of Proposition 2.5.4. Let $c \in H^n_c(U)$ be a cohomology class. Then there exists a finite collection $\mathcal{B}$ of Euclidean balls compactly contained in U for which c is in the image of τ_{UV}, where $V = \bigcup \mathcal{B}$. Let B' be a Euclidean ball compactly contained in V. By Lemma 2.5.7, c is in the image of $\tau_{UB'}$. Thus, by Lemma 2.5.6, c is in the image of τ_{UB}. □

2.6 Excursion: Cohomological dimension and separation in Euclidean spaces

As an application of Theorem 2.1.1, we consider cohomological dimension of closed sets in Euclidean spaces. The defintion of the cohomological dimension reads as follows.

Definition 2.6.1. The *cohomological dimension* $\dim_{\mathbb{Z}} X$ *of space* X *is at most* n if $H_c^k(U) = 0$ for all $k > n$ and each open set $U \subset X$. The space X has cohomological dimension n if it has cohomological dimension at most n and it does not have cohomological dimension at most $(n-1)$.

Clearly, the Euclidean space $\mathbb{R}^n$ and all its open subsets have cohomological dimension n. Similarly, closed sets having non-empty interior have cohomological dimension n. Closed sets with empty interior, on the other hand, have cohomological dimension at most $n-1$. We record this as a lemma.

Lemma 2.6.2. *Let $A \subset \mathbb{R}^n$ be a closed set having empty interior and $U \subset A$ open in A. Then $H_c^k(U) = 0$ for all $k > n-1$. In particular,* $\dim_{\mathbb{Z}} A \le n-1$.

Proof. We may assume that U is connected. Let $V \subset \mathbb{R}^n$ be a connected open set in $\mathbb{R}^n$ for which $V \cap A = U$. Since $V \cap A$ is closed in V and $V \cap A$ is a proper subset of V, we have, by Corollary 2.5.2, that $H^k(V \cap A) = 0$ for $k > n-1$. The claim follows. □

Closed codimension 1 subsets in $\mathbb{R}^n$ have an interesting elementary characterization. They are exactly the sets that locally separate $\mathbb{R}^n$. More formally, we have the following definition and theorem.

Definition 2.6.3. A subset $E \subset X$ *separates* X *locally around* $x \in X$ if for each neighborhood U of x there exists a connected neighborhood $V \subset U$ of x for which $V \setminus E$ is not connected. A subset E *separates* X *locally* if E separates X locally at some point $x \in X$.

Theorem 2.6.4. *Let $A \subset \mathbb{R}^n$ be a closed subset having empty interior. Then* $\dim_{\mathbb{Z}} A = n-1$ *if and only if A separates $\mathbb{R}^n$ locally.*

Proof. Note that, since A has empty interior, $\dim_Z A \le n-1$. Thus we consider two cases: first that $\dim_{\mathbb{Z}} A = n-1$ and then $\dim_{\mathbb{Z}} A < n-1$

Suppose first that $\dim_{\mathbb{Z}} A < n-1$. Let V be a domain in $\mathbb{R}^n$. Then $U = V \cap A$ is open in A and $H_c^{n-1}(U) = 0$ by the dimension assumption. By the exactness of the sequence

$$H_c^{n-1}(U) \xrightarrow{\partial} H_c^n(V \setminus A) \xrightarrow{\tau} H_c^n(V) \xrightarrow{\iota^*} H_c^n(U), \tag{2.6.1}$$

the push-forward $\tau\colon H_c^n(V \setminus A) \to H_c^n(V)$ is an isomorphism. Hence $V \setminus A$ is connected. Hence A does not locally separate $\mathbb{R}^n$.

Suppose now that $\dim_{\mathbb{Z}} A = n-1$. Then there exists a domain U in A for which $H_c^{n-1}(U) \ne 0$. Let V be a domain in $\mathbb{R}^n$ for which $V \cap A = U$. Then (again) by the exactness of (2.6.1), we have that $H_c^n(V \setminus A) \not\cong \mathbb{Z}$. Thus $V \setminus A$ is not connected. Let V' be a component of $V \setminus A$ and $V'' = V \setminus \overline{V'}$. Since $\partial V' \cap V \subset A$ and A has empty interior, we conclude that $V'' \ne \emptyset$. Let $x \in \overline{V} \cap \overline{V''} \cap V$. Then $x \in A$.

We show that A separates $\mathbb{R}^n$ locally at x. Suppose towards contradiction that A does not locally separate $\mathbb{R}^n$ at x. Let B_x a Euclidean ball centered at x and contained in V. By assumption, $B_x \setminus A$ is connected. Since $(B_x \setminus A) \cap V' = B_x \cap V' \neq \emptyset$, we have that $B_x \setminus A \subset V'$ by connectedness. This is a contradiction, since $(B_x \setminus A) \cap V'' = B_x \cap V'' \neq \emptyset$. Thus A separates $\mathbb{R}^n$ locally at x. $\square$

Chapter 3

Degree

We recall from Chapter 2 that, given a domain $U \subset \mathbb{R}^n$, the choice of a generator of $H^n_c(U;\mathbb{Z})$ is called an *orientation (class) of U*. In this section, we assume that we have fixed an orientation class $c_{\mathbb{R}^n} \in H^n_c(\mathbb{R}^n)$ and, for each domain $U \subset \mathbb{R}^n$, an orientation $c_U \in H^n_c(U)$ satisfying $c_{\mathbb{R}^n} = \tau_{\mathbb{R}^n U} c_U$.

Remark 3.0.5. *From the point of view of definitions, it suffices to fix an orientation class for each domain in $\mathbb{R}^n$. However, in proofs, we use repeatedly use the fact that, for a subdomain V of a domain U, we have the relation $\tau_{UV} c_V = c_U$.*

3.1 Global degree

Since a proper mapping $U \to V$ induces a pull-back homomorphism $H^n_c(V) \to H^n_c(U)$ and groups $H^n_c(U)$ and $H^n_c(V)$ are infinite cyclic, we may give the following definition.

Definition 3.1.1. The degree $\deg f$ of a proper mapping $f\colon U \to V$ between domains in $\mathbb{R}^n$ is the (unique) integer $\lambda \in \mathbb{Z}$ satisfying

$$f^* c_V = \lambda c_U.$$

We say that a proper mapping $f\colon U \to V$ is *orientation preserving* if $\deg f \geq 1$, and *orientation reversing* if $\deg f \leq -1$. Note that, a proper mapping f is neither orientation preserving nor orientation reversing if $\deg f = 0$. Note also that, since a homeomorphism has an inverse, it has either degree 1 or -1.

Since the composition of proper mappings is proper, we have the following product rule for the degree.

Lemma 3.1.2. *Let $f\colon U \to V$ and $g\colon V \to W$ be proper mappings between domains in $\mathbb{R}^n$. Then* $\deg(g \circ f) = \deg(g)\deg(f)$.

Proof. Since

$$\begin{aligned}\deg(g \circ f)c_U &= (g \circ f)^* c_W = f^* g^* c_W = f^* \left((\deg g) c_V\right) \\ &= (\deg g) f^* c_V = (\deg g)(\deg f) c_U,\end{aligned}$$

the claim follows. □

A similarly easy observation is that, since properly homotopic maps induce the same pull-back in cohomology, they have the same degree. We record this as an observation.

Observation 3.1.3. *Let $f\colon U \to V$ and $g\colon U \to V$ be properly homotopic mappings. Then* $\deg f = \deg g$.

3.2 Local degree

The definition of the local degree is based on the notion of an admissible domain.

Definition 3.2.1. Let $f\colon X \to Y$ be a map and $\Omega \subset X$ a pre-compact domain. A domain $W \subset Y$ is (f,Ω)-*admissible* if $W \subset Y \setminus f(\partial\Omega)$. A point $y \in Y$ is (f,Ω)-*admissible* if $y \notin f(\partial\Omega)$.

Remark 3.2.2. *Although it is not emphasized in the definition, given a pre-compact domain Ω, we are mostly interested in an (f,Ω)-admissible domain contained in $f\Omega$. Hence it is typical to assume, in the context of the local degree, that the mapping f is, in addition, an open map. Formally, of course this is not necessary.*

The fundamental role of the admissible domains is highlighted by the following lemma, which states that restrictions to admissible domains are proper mappings.

Lemma 3.2.3. *Let $f\colon X \to Y$ be a map, $\Omega \subset X$ a pre-compact domain, and let $W \subset Y$ be an (f,Ω)-admissible domain. Then the restriction $f|_{f^{-1}W\cap\Omega}\colon f^{-1}W \cap \Omega \to W$ is a proper mapping.*

Proof. Let $E \subset W$ be a compact set. Since Ω is pre-compact, it suffices to show that $f^{-1}E \cap \Omega$ is closed in X. Since W is (f,Ω)-admissible, we have that $f^{-1}W \cap \partial\Omega = \emptyset$, and, in particular, $f^{-1}E \cap \partial\Omega = \emptyset$. Thus $f^{-1}E \cap \Omega = f^{-1}E \cap \overline{\Omega}$ is closed. The claim follows. □

To simplify notation, given a pre-compact domain $\Omega \subset X$ and an (f,Ω)-admissible domain $W \subset Y$, we denote $D(\Omega, f, W) \subset X$ the pre-image $f^{-1}W \cap \Omega$. Note that, $D(\Omega, f, W)$ may be empty. In that case, the restriction of f to $D(\Omega, f, W)$ is the so-called *empty map*.

Having Lemma 3.2.3 at our disposal, we are ready to give the definition of the local degree of a mapping with respect to an admissible domain.

Definition 3.2.4. The *local degree* $\deg(\Omega, f, W)$ *of a map* $f\colon U \to V$ *between Euclidean domains with respect to a pre-compact domain* $\Omega \subset U$ *and an* (f,Ω)*-admissible domain* $W \subset V$ is the integer $\lambda \in \mathbb{Z}$ satisfying

$$(\tau_{\Omega D(\Omega,f,W)} \circ (f|_{D(\Omega,f,W)})^*)\, c_W = \lambda c_\Omega. \tag{3.2.1}$$

Remark 3.2.5. *This cumbersome definition is more reasonable after observing that the open set* $D(\Omega, f, W)$ *need not be connected, and hence the group* $H^n_c(D(\Omega, f, W))$ *is a priori only a direct sum of infinite cyclic groups. Consider for examplke the mapping* $z \mapsto z^2$ *for* $\Omega = B^2(0,1)$ *and* $W = B^2(1/2, 1/4)$*. Heuristically, the role of the push-forward* $\tau_{\Omega D(\Omega,f,W)}$ *in* (3.2.1) *is to sum up the degrees of mappings* $f|_D\colon D \to V$ *for different components* D *of* $D(\Omega, f, W)$*.*

We proceed now to define the local degree at an admissible point. For this reason we show that the local degree $\deg(\Omega, f, W)$ depends only on the component $Y \setminus f(\partial\Omega)$ containing W. We formulate this result as follows.

Proposition 3.2.6. *Let* $f\colon U \to V$ *be a map,* $\Omega \subset U$ *a pre-compact domain in* U*, and let* $W_1 \subset W_2 \subset f\Omega \setminus f(\partial\Omega)$ *be* (f,Ω)*-admissible domains. Then*

$$\deg(\Omega, f, W_1) = \deg(\Omega, f, W_2).$$

Proof. Let $W = W_1 \cap W_2$. Then W is an (f,Ω) admissible domain, and it suffices to show that $\deg(\Omega, f, W_i) = \deg(\Omega, f, W)$ for $i = 1, 2$. Let $D = f^{-1}W \cap \Omega$ and $D_i = f^{-1}W_i \cap \Omega$ for $i = 1, 2$. Then $f|_D = (f_{D_i})|_D$ and, by Lemma 1.5.16,

$$\tau_{U_i U}(f|_D)^* c_W = (f|_{D_i})^* \tau_{W_i W}(c_W) = (f|_{D_i})^* c_{W_i}.$$

for $i = 1, 2$. Hence

$$\begin{aligned} \deg(\Omega, f, W) c_\Omega &= \tau_{\Omega D}(f|_D)^* c_W \\ &= \tau_{\Omega D_i}\left(\tau_{D_i D}(f|_{D_i})^* c_{W_i}\right) \\ &= \tau_{\Omega D_i}(f|_{D_i})^* c_{D_i} = \deg(\Omega, f, W_i) c_\Omega \end{aligned}$$

for $i = 1, 2$. The claim follows. □

Having Proposition 3.2.6 at our disposal, we may define the local degree at an admissible point.

Definition 3.2.7. Let $f\colon U \to V$ be a map between domains in $\mathbb{R}^n$ and $\Omega \subset U$ a pre-compact domain. Given an (f,Ω)-admissible point $y \in Y$, the *local degree* $\deg(\Omega, f, y) \in \mathbb{Z}$ *of* f *at* y *with respect to* Ω is the integer satisfying

$$\deg(\Omega, f, y) = \deg(\Omega, f, W),$$

for any (f,Ω)-admissible domain $W \subset Y$ containing y.

Remark 3.2.8. *If an (f,Ω)-admissible point $y \in V$ is not in the image of Ω, then $D(\Omega, f, W) = \Omega \cap f^{-1}W = \emptyset$ for the component W of $V \setminus f(\partial\Omega)$. Thus $H^n_c(D(\Omega, f, W)) = 0$ and $\deg(\Omega, f, y) = 0$.*

We finish this section with a trivial (but important) observation that the local degree is locally constant at admissible points.

Lemma 3.2.9. *Let $f\colon U \to V$ be a map, $\Omega \subset U$ a pre-compact domain. Then the function $\deg(\Omega, f, \cdot)\colon V \setminus f(\partial\Omega) \to \mathbb{Z}$,*

$$y \mapsto \deg(\Omega, f, y),$$

is locally constant.

Proof. Let W be a component of $V \setminus f(\partial\Omega)$ and $y_1, y_2 \in W$. Then

$$\deg(\Omega, f, y_1) = \deg(\Omega, f, W) = \deg(\Omega, f, y_2)$$

by definition. The claim follows. □

3.3 Local index of discrete and open maps

In this section we add more assuptions to our mapping in order to define the local index at a point. The main difference to the previous discussion is that the local index is defined in the domain of the mapping instead of target and that there is no need to consider admissibility of the point– the index can be defined at each point of the domain. To define the local index, we consider first the notion of normal domains and neighborhoods.

For the basis of the discussion, we recall a simple lemma.

Lemma 3.3.1. *Let $f\colon U \to V$ be an open mapping between Euclidean domains and let $\Omega \subset U$ be a pre-compact domain. Then $\partial f\Omega \subset f(\partial\Omega)$.*

Proof. Since Ω is pre-compact, $f(\overline{\Omega})$ is compact and hence closed. Thus

$$\partial f(\Omega) \subset \overline{f(\Omega)} \subset f(\overline{\Omega}) = f(\Omega) \cup f(\partial\Omega).$$

It suffices to show that $\partial f(\Omega) \cap f\Omega = \emptyset$. Suppose $y \in \partial f(\Omega) \cap f\Omega$. Then there exists $x \in \Omega$ for which $f(x) = y$. Since f is an open map and Ω is open, we conclude that $y = f(x)$ is an interior point of $f\Omega$. This contradicts the assumption $y \in \partial f(\Omega)$. The claim follows. □

It not however, true in general that $\partial f\Omega = f(\partial\Omega)$; take for example the mapping $z \mapsto z^3$ in the complex plane $\mathbb{C}$ and consider the upper half disk $\Omega = \{x + iy\colon x^2 + y^2 < 1,\ y > 0\}$. The domains which satisfy the condition $\partial f\Omega = f\partial\Omega$ are called normal domains.

Definition 3.3.2. Let $f\colon X \to Y$ be an open mapping. A pre-compact domain Ω is a *normal domain (for the mapping) f* if $f(\partial\Omega) = \partial f\Omega$.

Normal domains exist in abundance.

Lemma 3.3.3. *Let $f\colon U \to V$ be an open mapping between Euclidean domains, Ω a pre-compact domain, and let W be a component of $f\Omega \setminus f(\partial\Omega)$. Then each component D of $f^{-1}W \cap \Omega$ is a normal domain.*

Proof. Since Ω is pre-compact, we have that D is pre-compact and $\partial fD \subset f(\partial D)$. Thus it remains to verify that $f(\partial D) \subset \partial fD$.

Let $x \in \partial D$. Then $f(x) \in \overline{fD}$. Suppose $f(x) \in fD$. Then, by continuity, there exists a connected neighborhood G of x contained in $f^{-1}fD = f^{-1}W$. Since D is a component of $f^{-1}W$, we conclude that x is an interior point of D. This is contradiction. Thus $f(x) \notin fD$ and $f(x) \in \partial fD$. □

The restriction of an open mapping to a normal domain is a proper and closed map.

Lemma 3.3.4. *Let $f\colon U \to V$ be an open map and W a normal domain in U. Then $f|_W\colon W \to fW$ is a proper and closed map.*

Proof. We verify first that $f|_W$ is proper. Let $E \subset fW$ be a compact set. Since E is compact and fW is open, we have that $\partial fW \cap E = \emptyset$. Since W is a normal domain, $f^{-1}E \cap \partial W = \emptyset$. Since $f^{-1}E$ is closed in U and $f^{-1}E \cap W = f^{-1}E \cap (W \cup \partial W) = f^{-1}E \cap \overline{W}$, we conclude that $(f|_W)^{-1}E = f^{-1}E \cap W$ is closed in W. Since W is pre-compact, $(f|_W)^{-1}E$ is compact.

Suppose now that $A \subset W$ is a closed set in W. Since W is pre-compact, $\overline{A}$ is compact and $f\overline{A}$ is closed. Thus $f\overline{A} = \overline{fA}$ by continuity and the definition of the closure. Since $\overline{A} \setminus A \subset \partial W$ and $f(\overline{A} \cap \partial W) \subset \partial fW$, we have that

$$\overline{fA} \cap fW = f\overline{A} \cap fW = (fA \cap fW) \cup \big(f(\overline{A} \setminus A) \cap fW\big) = fA.$$

Thus fA is closed in fW. The claim follows. □

We obtain local surjectivity of the restrictions of open maps to normal domains as a corollary. We record this observation as follows.

Corollary 3.3.5. *Let $f\colon U \to V$ be an open mapping between Euclidean domains, $\Omega \subset U$ a pre-compact domain, and $D \subset fU \setminus f(\partial\Omega)$ a domain. Then, for each component W of $f^{-1}D \cap \Omega$, the restriction $f|_W\colon W \to D$ is surjective.*

Proof. By Lemma 3.3.3, W is a normal domain and by Lemma 3.3.4 the restriction $f|_W$ is a closed map. Thus fW is open and closed in D. Since D is connected, we conclude that $D = fW$. □

To define the local index of a discrete and open mapping, we change now our point of view and discuss normal neighborhoods of points.

Definition 3.3.6. Let $f\colon X \to Y$ be a discrete and open map and let $x \in X$. A pre-compact domain $\Omega \subset X$ is a *normal neighborhood of x (with respect to mapping f* if Ω is a normal domain for f and $f^{-1}(f(x)) \cap \overline{\Omega} = \{x\}$.

Remark 3.3.7. *The relation of normal neighborhoods and admissible points is obvious. For a normal neigbhborhood Ω of x we have $f(x) \notin f(\partial\Omega)$. Thus $f(x)$ is an (f,Ω)-admissible point.*

Since the existence of a normal neighborhood implies discreteness of the pre-image fiber at that point, it is natural to consider the existence of normal neighborhoods for points under the assumptions that the mapping is both discrete and open. Similarly as normal domains for open mappings also the normal neighborhoods exist in abundance for discrete and open maps.

Lemma 3.3.8. *Let $f\colon U \to V$ be a discrete and open map, $x \in X$, and Ω a pre-compact neighborhood of x in U. Then there exists a normal neighborhood $W \subset \Omega$ of x. Moreover, if $D \subset fW$ is a neighborhood of $f(x)$ then $f^{-1}D \cap W$ is a normal neighborhood of x.*

Proof. Since f is discrete, there exists a neighborhood D of x, compactly contained in Ω, for which $f^{-1}(f(x)) \cap \overline{D} = \{x\}$. Since $f(x) \notin f(\partial D)$, there exists a component V of $fD \setminus f(\partial D)$ containing $f(x)$. Let now W be the component of $f^{-1}V$ containing x. Then W is a normal domain by Lemma 3.3.3 and satisfies $f^{-1}f(x) \cap \overline{W} = \{x\}$.

For the second statement, let $D \subset fW$ be a connected neighborhood of $f(x)$. Since W is a normal domain, $D \cap f(\partial\Omega) = \emptyset$. Hence, by Corollary 3.3.5, the restriction $f|_{f^{-1}D\cap W}\colon f^{-1}D \cap W \to D$ maps the components of $f^{-1}D$ map surjectively on D. Since $f^{-1}f(x) \cap W = \{x\}$, we conclude that $f^{-1}D$ is connected. Hence $f^{-1}D \cap W$ is a normal neighborhood of x. □

The local index of a discrete and open map at a point is the local degree of the restriction of the map to a normal neighborhood. The following lemma shows that the local index defined this way is well-defined.

Lemma 3.3.9. *Let $f\colon U \to V$ be a discrete and open map between Euclidean domains and $x \in U$. Suppose $\Omega_1 \subset \Omega_2$ are normal neighborhoods of x. Then*

$$\deg(\Omega_1, f, f(x)) = \deg(\Omega_2, f, f(x)).$$

Proof. We observe first that, by Lemma 3.3.8, $f^{-1}f\Omega_1 \cap \Omega_2$ is a domain. Since $\Omega_1 \subset f^{-1}f\Omega_1$ and Ω_1 is a normal domain, we conclude that $f^{-1}f\Omega_1 \cap \Omega_2 = \Omega_1$. Let now $W = f\Omega_1$.

Since the restrictions $f_i = f|_{f^{-1}W\cap\Omega_i}\colon f^{-1}W \cap \Omega_i \to W$ for $i = 1, 2$ are the same map and, in particular, $f_2^* c_W = \tau_{\Omega_1,\Omega} f_1^* c_W$, where $\Omega = f^{-1}W \cap \Omega_i$, we have

$$\begin{aligned}
\deg(\Omega_2, f, f(x))c_{\Omega_2} &= \deg(\Omega_2, f, W)c_{\Omega_2} = \tau_{\Omega_2\Omega}(f_2)^* c_W \\
&= \tau_{\Omega_2\Omega_1}\left(\tau_{\Omega_1\Omega}(f_1)^* c_W\right) = \tau_{\Omega_2\Omega_1}(\deg(\Omega_1, f, W)c_{\Omega_1}) \\
&= \deg(\Omega_1, f, W)c_{\Omega_2} = \deg(\Omega_1, f, f(x))c_{\Omega_2}.
\end{aligned}$$

The proof is complete. □

Definition 3.3.10. Let $f\colon U \to V$ be a discrete and open map between domains in $\mathbb{R}^n$. The *local index* $i(x, f)$ *of* f *at* $x \in U$ is the integer satisfying

$$i(x, f) = \deg(\Omega, f, f(x)),$$

where Ω is a normal neighborhood of x.

The local index of the map is merely the degree of a restriction of the map to a normal neighborhood.

Lemma 3.3.11. *Let $f\colon U \to V$ be a discrete and open map between Euclidean domains, $x \in U$, and Ω a normal neighborhood of x in U. Then*

$$i(x, f) = \deg(f|_\Omega\colon \Omega \to f\Omega).$$

Proof. Since Ω is a normal neighborhood, $f\Omega$ is (trivially) an (f, Ω)-admissible domain and $D(\Omega, f, f\Omega) = \Omega$. Hence $\deg(\Omega, f, f\Omega) = \deg(f|_\Omega\colon \Omega \to f\Omega)$. Thus

$$\deg(\Omega, f, f(x)) = \deg(\Omega, f, f\Omega) = \deg(f|_\Omega\colon \Omega \to f\Omega).$$

The claim follows. □

As an immediate consequence we obtain a useful product formula for the local index.

Corollary 3.3.12. *Let $f\colon U \to V$ and $g\colon V \to W$ be discrete and open mappings between Euclidean domains. Then, for each $x \in U$,*

$$i(x, g \circ f) = i(f(x), g)i(x, f).$$

Proof. Let Ω be normal neighborhood of x with respect to the mapping $g \circ f$ and let $\Omega' = f(\Omega)$. Then Ω is a normal neighborhood of x with respect to the mapping f and $\Omega' = f\Omega$ is a normal neighborhood of $f(x)$ with respect to g. Thus

$$\begin{aligned}
i(x, g \circ f) &= \deg((g \circ f)|_\Omega\colon \Omega \to (g \circ f)\Omega) \\
&= \deg(g|_{f\Omega}\colon f\Omega \to (g \circ f)\Omega) \deg(f|_\Omega\colon \Omega \to f\Omega) \\
&= i(f(x), g)i(x, f).
\end{aligned}$$

□

We also note, as an observation, that the local index of a local homeomorphism is locally constant.

Lemma 3.3.13. *Let $f\colon U \to V$ be a local homeomorphism between domains in $\mathbb{R}^n$. Then the function $x \mapsto i(x, f)$ is either the constant function* 1 *or* -1.

Proof. Let $x \in U$. Since f is a local homeomorphism, there exists a normal neighborhood W of x so that $f|_W\colon W \to fW$ is a homeomorphism. Since, for each $x' \in W$, $i(x', f) = \deg(W, f, fW) = i(x, f)$, the function $x \mapsto i(x, f)$ is locally constant. Moreover, since f is a local homeomorphism, $i(x, f) = \deg(W, f, fW) = \pm 1$ at each $x \in U$. □

We finish this section, and the discussion on the degree theory, with the summation formula for the local index.

Theorem 3.3.14. *Let $f\colon U \to V$ be a discrete and open map, $\Omega \subset U$ a pre-compact domain, and $y \in V \setminus f(\partial\Omega)$. Then*

$$\sum_{x \in f^{-1}(y) \cap \Omega} i(x, f) = \deg(\Omega, f, y).$$

Remark 3.3.15. *We would like to note that this summation formula gives a simple method to calculate a local (but also global) degree of a discrete and open map. The reader may want to consider maps $z \mapsto z^k$, $re^{i\pi t} \mapsto re^{i\pi k t}$, and their products as examples. Or piece-wise linear maps between manifolds.*

Proof of Theorem 3.3.14. Since f is discrete and $\overline{\Omega}$ is compact, the set $f^{-1}(y)$ is finite. Let $\{x_1, \ldots, x_k\} = f^{-1}(y)$. For each $j \in \{1, \ldots, k\}$, we fix a normal neighborhood $\Omega_j \subset \Omega$ of x_j having the property that the sets $\overline{\Omega_j}$ are pair-wise disjoint. Let $W \subset V$ be a neighborhood of y which is (f, Ω_j)-admissible for each $j = 1, \ldots, k$, i.e. $W \subset V \setminus \bigcup_{j=1}^k f(\partial\Omega_j)$. Let also $W_j' = f^{-1}W \cap \Omega_j$ and $f_j = f|_{W_j'}\colon W_j' \to W$ for each j.Finally, let $W' = f^{-1}W \cap \Omega$. By Corollary 3.3.5, each component of $f^{-1}W \cap \Omega$ maps surjectively on W. Thus $W' = \bigcup_{j=1}^k W_j'$.

Having these notations at our disposal, we have

$$\sum_{x \in f^{-1}(y)} i(x, f) = \sum_{j=1}^k \deg(\Omega_j, f, y) = \sum_{j=1}^k \deg(\Omega_j, f, W)$$

by the definition of local degree at a point.

By the definition of local degree, we have

$$\begin{aligned}\sum_{j=1}^{k}\deg(\Omega_j,f,W)c_\Omega &= \sum_{j=1}^{k}\tau_{\Omega\Omega_j}\left(\deg(\Omega_j,f,W)c_{\Omega_j}\right)\\ &= \sum_{j=1}^{k}\tau_{\Omega\Omega_j}\left(\tau_{\Omega_j W'_j}f_j^* c_W\right)\\ &= \sum_{j=1}^{k}\tau_{\Omega W'_j}f_j^* c_W = \tau_{\Omega W'}\left(\sum_{j=1}^{k}\tau_{W'W'_j}f_j^* c_W\right).\end{aligned}$$

Since W' is a disjoint union of domains $W'_1,\dots,W'_k$, we have, by Theorem 1.7.3, that the homomorphism $J\colon \bigoplus_{j=1}^k H_c^n(W'_j)\to H_c^n(W')$, $(c_j)\mapsto \sum_{j=1}^k \tau_{W'W'_j}c_j$, is an isomorphism and the homomorphism $I\colon H_c^n(W')\to \bigoplus_{j=1}^k H_c^n(W'_j)$, $c\mapsto(\iota^*_{W'W'_j}c)$, is its inverse. Since $f|_{W'}\circ\iota_{W'_jW'}=f|_{W'_j}\colon W'_j\to W$, we have $\iota^*_{W'_jW'}\circ(f|_{W'})^*=f_j^*$ for each $j=1,\dots,k$. Thus

$$\begin{aligned}\sum_{j=1}^{k}\tau_{W'W'_j}f_j^* c_W &= J(f_1^*c_W,\dots,f_k^*c_W)\\ &= J(\iota^*_{W'_1W'}(f|_{W'})^*c_W,\dots,\iota^*_{W'_kW'}(f|_{W'})^*c_W)\\ &= (J\circ I)((f|_{W'})^*c_W)=(f|_{W'})^*c_W.\end{aligned}$$

We conclude that

$$\sum_{j=1}^{k}\deg(\Omega_j,f,y)c_\Omega=\tau_{\Omega W'}(f|_{W'})^*c_W=\deg(\Omega,f,W)c_\Omega=\deg(\Omega,f,y)c_\Omega.$$

The proof is complete. □

Chapter 4

Väisälä's theorem

4.1 Statement

We prove the following version of Väisälä's theorem. This version is equivalent to the version in the introduction if we take as granted that discrete and open sets preserve the cohomological dimension. See Borel [Bor60] and Engelking [Eng78] for discussion on various definitions of dimension and e.g. Church–Hemmingsen [CH60] for discussion on the mappings and dimension.

We recall two definitions before the statement of Väisälä's theorem.

Definition 4.1.1. A discrete and open map $f\colon X \to Y$ is a *branched cover.* The *branch set* B_f *of a branced cover* is the set

$$B_f = \{x \in X \colon f \text{ is not a local homeomorphism at } x\}.$$

Definition 4.1.2. A subset $A \subset X$ *separates* X *locally at* $x \in X$ if there exists a neighborhood U of x so that, for each neighborhood $V \subset U$ of x, the set $V \setminus A$ is not connected.

Theorem 4.1.3. *Let $f\colon U \to V$ be a discrete and open map between open sets in $\mathbb{R}^n$. Then the branch set B_f has no interior and does not locally separate U at any point.*

The proof is in two steps. In the first step we show that $\mathrm{int} B_f = \emptyset$. This is the easier part of the proof. In the second step we show that B_f does not locally separate U. This is harder and we use all the theory we have developed. In the course of the proof of Theorem 4.1.3, we show that fB_f has no interior (Theorem 4.2.5), and hence also $f^{-1}fB_f$ has no interior, since f is open. We are not aware of purely elementary proof for the fact that fB_f does not locally separate V, and we do not discuss this here.

Before discussing the proof, we record an important corollary of Väisälä's theorem.

Theorem 4.1.4. *Let $f: U \to V$ be a discrete and open map between open sets in $\mathbb{R}^n$. Let $x \in U$ be a point and U_x a normal neighborhood of x in U. Then, for each $z \in U_x$, the local indices $i(z,f)$ and $i(x,f)$ have the same sign and*

$$|i(x,f)| = \max_{x' \in U_x} \# f^{-1}f(x') \geq |i(z,f)|.$$

Proof. Let $z \in U_x$ and let U_z be a normal neighborhood of z contained in U_x. Let $x' \in U_z \setminus f^{-1}fB_f$ be a point; note that $f^{-1}fB_f$ has empty interior. Then, by Lemma 3.2.9 and the summation theorem (Theorem 3.3.14),

$$i(x,f) = \deg(U_x, f, f(x)) = \deg(U_x, f, f(x')) = \sum_{x'' \in f^{-1}(x') \cap U_x} i(x'', f).$$

and

$$i(z,f) = \deg(U_z, f, f(z)) = \deg(U_z, f, f(x')) = \sum_{x'' \in f^{-1}(x') \cap U_z} i(x'', f).$$

Since $U_x \setminus B_f$ is connected by Väisälä's theorem, the function $x' \mapsto i(x', f)$ is constant in $U_f \setminus B_f$ by Lemma 3.3.13. Thus $i(x,f)$ and $i(z,f)$ have the same sign. Moreover, $|i(x,f)| \geq |i(z,f)|$.

Since f is a local homeomorphism at each point $f^{-1}f(x') \cap U_x$, the remaining claim $|i(x,f)| = \max_{x' \in U_x} \# f^{-1}f(x')$ follows now from the observation that $|i(x'', f)| = 1$ for each $x'' \in f^{-1}f(x') \cap U_x$. □

Regarding local separation, we have the following general lemma.

4.2 The branch set has no interior

We begin the proof of Väisälä's theorem by showing that the branch set has no interior. We formulate this result as a theorem in the case of n-manifolds.

Theorem 4.2.1. *Let $f: M \to N$ be a discrete and open map between n-manifolds. Then $\mathrm{int} B_f = \emptyset$.*

For the proof we recall two general observations.

Observation 4.2.2. *The branch set of discrete and open map is a closed set.*

Remark 4.2.3. *Note that, the image of the branch set need not be closed. This is however the case if the mapping is, in addition, proper.*

Observation 4.2.4. *Let $f: U \to V$ be a discrete and open map between open sets in $\mathbb{R}^n$, and let $D \subset M$ be an open set. Then $f|_D: D \to V$ is discrete and open, and $B_{f|_D} = B_f \cap D$.*

Proof of Theorem 4.2.1. Suppose $\mathrm{int} B_f \neq \emptyset$. Then there exists a pre-compact open set $U \subset B_f$. Hence $g = f|_U : U \to fU$ is a discrete and open map so that $\# g^{-1}(y) < \infty$ for each $y \in fU$. For each $k \geq 0$, let $M_k = \{x \in U : \# g^{-1}g(x) \leq k\}$. Then $\bigcup_{k\geq 0} M_k = U$ and, by the Baire category theorem[1], there exists smallest $k_0 \in \mathbb{N}$ for which $\mathrm{int} M_{k_0} \neq \emptyset$.

Let $V = \mathrm{int} M_{k_0}$, $x_1 \in V$, and consider the restriction $g|_V : V \to gV$. To complete the proof it suffices to show that $g|_V$ is a local homeomorphism at x_1. Indeed, since $g|_V = f|_V$, V is open in U, we have that f is a local homeomorphism at x_1. This is a contradiction, since $x_1 \in B_f$.

Let $\{x_1, x_2, \ldots, x_{k_0}\} = g^{-1}g(x_1)$. For each $j = 1, \ldots, k_0$, we fix a neighborhood U_j of x_j satisfying $U_j \cap U_i = \emptyset$ for each $j \neq i$. Then $\bigcap_j fU_j$ is a neighborhood of $g(x_1)$ and $W = U_1 \cap g^{-1}(\bigcap_j fU_j)$ is a neighborhood of x_1. We show that the restriction $g|_W$ is injective. Let $x \in W$. Then $\# g^{-1}g(x) \cap U_j \geq 1$ for each $j = 1, \ldots, k_0$ and $\# g^{-1}(g(x)) = k_0$. Thus $g^{-1}g(x) \cap U_1 = \{x\}$. Since $g|_W$ is injective and open, we conclude that $g|_W : W \to gW$ is a homeomorphism. The proof is complete. □

Using the fact that B_f has empty interior, we obtain also that fB_f has empty interior. It is interesting that this result is not a trivial consequence of Theorem 4.2.1.[2]

Theorem 4.2.5. *Let $f : M \to N$ be a discrete and open map between n-manifolds. Then* $\mathrm{int} fB_f = \emptyset$.

Proof. Let $\Omega \subset M$ a pre-compact normal domain in M and $g = f|_\Omega : \Omega \to f\Omega$. We show first that $\mathrm{int} gB_g = \emptyset$.

Let $y \in gB_g$ and let G be a neighborhood of y. It suffices to show that $G \setminus fB_f \neq \emptyset$.

Since Ω is pre-compact, $g^{-1}(y) = \{x_1, \ldots, x_m\}$. Let $V \subset G$ be a domain for which the components $W_1, \ldots, W_m$ of $g^{-1}V$ are normal neighborhoods of points $x_1, \ldots, x_m$, respectively. Then the sets $W_1, \ldots, W_m$ are pair-wise disjoint and $gW_i = V$ for each $i = 1, \ldots, m$. Let $W_i' = W_i \setminus B_g$ for each $i = 1, \ldots, m$.

Clearly W_i' is open and dense in W_i. Further, since g is open, gW_i' is open. We show that gW_i' is also dense in V for each $i = 1, \ldots, m$.

Let $y' \in V$ and D be a neighborhood of y' in V. Let $i \in \{1, \ldots, m\}$. Since $g^{-1}D \cap W_i$ is open and $\mathrm{int} B_g = \emptyset$, there exists, $x_i' \in g^{-1}D \cap W_i'$. Thus $g(x_i') \in D$. Hence $gW_i' \cap D \neq \emptyset$. Thus gW_i' is dense in V.

Since each gW_i' is open and dense in gW_i, we have, by Baire's theorem, that $\bigcap_{i=1}^m gW_i'$ is dense in V. In particular, there exists $z \in \bigcap_{i=1}^m gW_i' \cap D$.

[1] By the *Baire category theorem* a locally compact Hausdorff space is not a countable union of nowhere dense closed sets.

[2] The argument given here, which nicely avoids the use of dimension theory, is due to Rami Luisto.

Since $g^{-1}D = \bigcup_{i=1}^{m}(g^{-1}D \cap W_i)$, we have

$$g^{-1}(z) \subset \bigcup_{i=1}^{m} W_i' = \bigcup_{i=1}^{m}(W_i \setminus B_g) \subset \Omega \setminus B_g$$

Thus $z \in D \setminus gB_g$. Since $D \subset G$, we conclude that $\mathrm{int} gB_g = \emptyset$.

We show now the general case. Since M is σ-compact, there exists a countable collection $\{\Omega_i\}_{i\geq 0}$ of pre-compact normal domains of f in M for which $B_f \subset \bigcup_i \Omega_i$. Let $g_i = f|_{\Omega_i} \colon \Omega_i \to f\Omega_i$. Then $fB_f = \bigcup_i g_i B_{g_i}$. Since each $g_i B_{g_i}$ is closed in $f\Omega_i$ and $\mathrm{int} g_i B_{g_i} = \emptyset$, we have, by Baire's theorem, that $\mathrm{int} fB_f = \emptyset$. □

4.3 The branch set does not separate locally

The proof of the remaining part of Theorem 4.1.3 is based on the idea that we may divide B_f into two parts: to points in which B_f locally separtes (bad part) and to points at which B_f does not locally separate (good part). Of course, the idea is to show that the bad part is empty. Having this idea in mind, we state first a general lemma regarding local separation.

Lemma 4.3.1. *Let X be a locally connected space, and $A \subset X$ a closed subset for which* $\mathrm{int} A = \emptyset$ *and $X \setminus A$ is not connected. Let*

$$F_A = \overline{\{x \in X \colon A \text{ separates } X \text{ locally at } x\}}.$$

Then $X \setminus F_A$ is not connected and $F_A \subset A$.

Proof. Since A is closed and X locally connected, $F_A \subset A$. Thus it suffices to prove that $X \setminus F_A$ is not connected.

Since $X \setminus A$ is not connected, there exist non-empty and pair-wise disjoint open sets U_1 and U_2 in X for which $X \setminus A = U_1 \cup U_2$. Let $V_i = (\mathrm{int}\overline{U_i}) \setminus F_A$ for $i = 1, 2$. Since $U_i \subset \mathrm{int}\overline{U_i}$ and $F_A \cap U_i = \emptyset$, we have $U_i \subset V_i$ for $i = 1, 2$.

We show now that $X \setminus F_A = V_1 \cup V_2$. Since $\mathrm{int} A = \emptyset$,

$$X = U_1 \cup U_2 \cup A = \overline{U_1} \cup \overline{U_2}.$$

Let $x \notin F_A$. Then there exists a connected neighborhood V of x contained in $X \setminus F_A$ for which $V \setminus A$ is connected. Thus either $V \setminus A \subset U_1$ or $V \setminus A \subset U_2$. Hence either $x \in \mathrm{int}\overline{U_1}$ or $x \in \mathrm{int}\overline{U_2}$. Thus $X \setminus F_A \subset V_1 \cup V_2$ and hence $X \setminus F_A = V_1 \cup V_2$.

To prove that V_1 and V_2 are pair-wise disjoint, we show first that $\partial U_1 \cap \partial U_2 \subset F_A$ for $i = 1, 2$. Let $x \in \partial U_1 \cap \partial U_2$, U a neighborhood of x in X, and let $V \subset U$ be a connected neighborhood of x. Then $V \cap U_i \neq \emptyset$ for $i = 1, 2$. Hence $V \setminus A = (V \cap U_1) \cup (V \cap U_2)$ is not connected. Thus $x \in F_A$.

Since

$$V_1 \cap V_2 = \left(\mathrm{int}\overline{U_1} \cap \mathrm{int}\overline{U_2}\right) \setminus F_A \subset (\partial U_1 \cap \partial U_2) \setminus F_A.$$

we have that $V_1 \cap V_2 = \emptyset$. We conclude that $X \setminus F_A$ is not connected. □

We record also a simple non-separation lemma for further use.

Lemma 4.3.2. *Let X be a locally connected space, $A \subset X$ a closed subset having empty interior and which does not locally separate X at any point. Then, for each domain U in X, the set $U \setminus A$ is connected.*

Proof. Suppose $U \setminus A$ is not connected. Let W be a component of $U \setminus A$ and $W' = U \setminus \overline{W}$. Note that $\overline{W} \cap \overline{W'} \subset A$ and $\overline{W} \cap \overline{W'} \neq \emptyset$.

Let $x \in (\partial W) \cap A$. Since x does not locally separate X, there exists a neighborhood U_x of x for which $U_x \setminus A$ is connected. Since $\mathrm{int} A = \emptyset$, we have that $U_x \cap W = \emptyset$ and $U_x \cap W' \neq \emptyset$. Since W is a component and $U_x \setminus A$ is connected, we have $(U_x \setminus A) \subset W$. This is a contradiction. Thus $U \setminus A$ is connected. □

We begin now the proof with an important observation, which ties maps of local degree 1 to homeomorphisms.[3]

Lemma 4.3.3. *Let $f \colon U \to V$ be a discrete and open map between domains in $\mathbb{R}^n$ Suppose that B_f does not locally separate U at any point, let Ω be normal domain in U for which $|\mu(\Omega, f, y)| = 1$ for each $y \in f\Omega$. Then $f|_\Omega \colon \Omega \to f\Omega$ is a homeomorphism.*

Proof. Since $f|_\Omega$ is a closed and surjective, it suffices to show that $f|_\Omega$ is injective.

Since B_f does not locally separate U at any point, we have, by Lemma 4.3.2, that $\Omega \setminus B_f$ is connected Thus, since $f|_{\Omega \setminus B_f}$ is a local homeomorphism, the function $\Omega \setminus B_f \to \mathbb{Z}$, $x \mapsto i(x, f)$, is either the constant function $x \mapsto 1$ or the function $x \mapsto -1$. Thus, for $y \in f\Omega \setminus (f|_\Omega) B_{f|_\Omega}$,

$$1 = |\mu(\Omega, f, y)| = \left| \sum_{x \in f^{-1}(y)} i(x, f) \right| = \# \left(f^{-1}(y) \cap \Omega \right)$$

Thus $f|_{\Omega \setminus B_f}$ is injective.

We show now that g is injective. Let x_1 and x_2 be points in Ω for which $g(x_1) = g(x_2)$, and let G_1 and G_2 be neighborhoods of x_1 and x_2, respectively. By Theorem 4.2.5, $(f|_\Omega) B_{f|_\Omega}$ has empty interior. Thus $(gG_1 \cap gG_2) \setminus gB_g \neq \emptyset$. By injectivity of $f|_{\Omega \setminus B_f}$, we have that $G_1 \cap G_2 \neq \emptyset$. Thus $x_1 = x_2$ and $f|_\Omega$ is injective. □

The key observation in the study of the bad part of B_f is to show that there is no *folding* in a generalized sense. The following argument is the key of the proof of Theorem 4.1.3.

[3]The statement can be read to say that *if B_f does not separate, there is no folding.*

Theorem 4.3.4 (Väisälä's no-reflection lemma)**.** *Let U be a domain in $\mathbb{R}^n$ and let U_1 and U_2 be pair-wise disjoint open subsets in U for which $\partial_U U_1 = \partial_U U_2$ and $\mathrm{cl}_U U_1 \cup \mathrm{cl}_U U_2 \neq U$. Then there is no homeomorphism $f\colon \mathrm{cl}_U U_1 \to \mathrm{cl}_U U_2$ for which $f|_{\partial_U U_1} = \mathrm{id}$.*

Proof. We begin with an observation. Let $\iota_i\colon U_i \to U_1 \cup U_2$ and $\bar{\iota}_i\colon \mathrm{cl}_U U_i \to \mathrm{cl}_U U_1 \cup \mathrm{cl}_U U_2$ be inclusions. Note that $\partial_U(U_1 \cup U_2) = \partial_U U_1 = \partial_U U_2$.

Let $\partial\colon H_c^{n-1}(\partial_U(U_1 \cup U_2)) \to H_c^n(U_1 \cup U_2)$ and $\partial_i\colon H_c^{n-1}(\partial_U U_i) \to H_c^n(U_i)$, for $i = 1, 2$, be the connecting homomorphisms in the exact sequences of pairs $(\mathrm{cl}_U(U_1 \cup U_2), \partial_U(U_1 \cup U_2))$ and $(\mathrm{cl}_U U_i, \partial_U U_i)$ for $i = 1, 2$, respectively. By Theorem 1.9.4, the diagram

$$\begin{array}{ccc}
H_c^{n-1}(\partial_U(U_1 \cup U_2)) & \xrightarrow{\partial} & H_c^n(U_1 \cup U_2) \\
\Vert & & \downarrow \iota_i^* \\
H_c^{n-1}(\partial_U U_i) & \xrightarrow{\partial_i} & H_c^{n-1}(U_i)
\end{array}$$

commutes, that is, $\partial_i = \iota_i^* \circ \partial$.

Suppose there exists a homeomorphism $f\colon \mathrm{cl}_U U_1 \to \mathrm{cl}_U U_2$ for which $f|_{\partial_U U_1} = \mathrm{id}$. Since $f(\partial_U U_1) = \partial_U U_2$, the restriction $g = f|_{U_1}\colon U_1 \to U_2$ is a homeomorphism. By Theorem 1.9.2, the diagram

$$\begin{array}{ccc}
H_c^{n-1}(\partial_U U_1) & \xrightarrow{\partial_1} & H_c^n(U_1) \\
\Vert \, (g|_{\partial_U U_1})^* & & \uparrow g^* \\
H_c^{n-1}(\partial_U U_2) & \xrightarrow{\partial_2} & H_c^n(U_2)
\end{array}$$

commutes. Thus $\partial_1 = g^* \circ \partial_2$.

Let now

$$I\colon H_c^n(U_1 \cup U_2) \to H_c^n(U_1) \oplus H_c^k(U_2)$$

be the isomorphism $c \mapsto (\iota_1^* c, \iota_2^* c)$. Then

$$I \circ \partial = (\iota_1^* \circ \partial, \iota_2^* \circ \partial) = (\partial_1, \partial_2) = (g^* \partial_2, \partial_2) = (g^*, \mathrm{id}) \circ \partial_2.$$

In particular, ∂ is not surjective. Indeed, let $a \in H_c^n(U_2)$ be a non-zero element and suppose that there exists $c \in H_c^{n-1}(\partial_U(U_1 \cup U_2))$ for which $(I \circ \partial)(c) = (0, a)$. Then

$$(0, a) = (g^* \partial_2 c, \partial_2 c) = (g^* a, a),$$

which is a contradiction, since g^* is an isomorphism. Hence $(0, a)$ is not in the image of $I \circ \partial$ and hence ∂ is not surjective.

Since $\mathrm{cl}_U(U_1 \cup U_2) \neq U$, we have $H_c^n(\mathrm{cl}_U(U_1 \cup U_2)) = 0$ by Corollary 2.5.2. Thus ∂ is surjective by exactness of the sequence

$$H_c^{n-1}(\partial_U(U_1 \cup U_2)) \xrightarrow{\partial} H_c^n(U_1 \cup U_2) \longrightarrow H_c^n(\mathrm{cl}_U(U_1 \cup U_2)).$$

This is a contradiction and the proof is complete. $\square$

Theorem 4.3.4 will be applied together with the following observation on local degree. We pass here from the setting of manifolds to domains in $\mathbb{R}^n$. We apply this lemma locally in the proof of Theorem 4.1.3.

Proposition 4.3.5. *Let $f\colon U \to V$ be a proper discrete and open map between domains in $\mathbb{R}^n$ and let $A \subset U$ be a closed set separating U and satisfying $\# f^{-1}f(x) = 1$ for each $x \in A$. Let Ω be a component of $U \setminus A$. Then Ω is a normal domain and* $\deg(\Omega, f, y) = \pm 1$ *for all $y \in f\Omega$.*

A bulk of the proof of this proposition is a verification of several properties of the mapping f. We separate these verifications as a lemma. The proof is similar to results related to normal domains. This is not a coincidence. Essentially we compensate the lack of pre-compacness of the domain Ω by the assumption that f is proper.

From now on, the closure and boundary of a subset is understood with respect to subdomains in $\mathbb{R}^n$, and not in terms of the ambient space $\mathbb{R}^n$. This is emphasized in the notation.

Lemma 4.3.6. *Let $f\colon U \to V$, $A \subset U$, and Ω be as in Proposition 4.3.5. Then $f|_\Omega\colon \Omega \to f\Omega$ is a proper and closed mapping for which $f\Omega$ is a component of $V \setminus fA$. Moreover, Ω is a normal domain and the restriction $h = f|_{\partial_U \Omega}\colon \partial_U \Omega \to \partial_V f\Omega$ is a well-defined homeomorphism.*

Proof. Since $f^{-1}fA = A$ by assumption, we have $f\Omega \subset V \setminus fA$. We show first that $f|_\Omega\colon \Omega \to f\Omega$ is proper. Let $E \subset f\Omega$ be a compact set. Then E is closed in $V \setminus fA$. Hence $f^{-1}E$ is closed in U and has empty intersection with A. Since Ω is closed in $U \setminus A$, we have that $f^{-1}E \cap \Omega$ is closed and hence compact. Thus $f|_\Omega\colon \Omega \to f\Omega$ is a proper map.

We also record at this stage also that $f|_\Omega$ is a closed map and that $f\Omega$ is a component of $V \setminus fA$. Indeed, let $E \subset \Omega$ be a closed set. Then $\mathrm{cl}_U E$ is closed in U. Let $E_1 \subset E_2 \subset \cdots$ be an exhaustion of $\mathrm{cl}_U E$ by compact sets. Since each fE_i is compact and f is proper, $f\mathrm{cl}_U E = \bigcup_i fE_i$ is closed. Thus $fE = f\mathrm{cl}_U E \cap f\Omega$ is closed in $f\Omega$. Since f is both open and closed, we conclude that $f\Omega$ is a component of $V \setminus fA$.

We observe also that $f(\partial_U \Omega) = \partial_V f\Omega$. Indeed, since f is open, we have $\partial_V f\Omega \subset f(\partial_U \Omega)$. On the other hand, since Ω is a component of $U \setminus A$, we have $\partial_U \Omega \subset A$. Thus $\Omega \cap f(\partial_U \Omega) \subset f\Omega \cap fA = \emptyset$. Hence $f(\partial_V \Omega) \subset \partial_V f\Omega$.

We show now that the restriction $h = f|_{\partial_U \Omega}\colon \partial_U \Omega \to \partial_V f\Omega$ is a homeomorphism. Since $\partial_U \Omega \subset A$ and $f|_A$ is injective, the map $h = f|_{\partial_U \Omega}\colon \partial_U \Omega \to \partial_V f\Omega$ is injective.

Since f is closed, we have $f\mathrm{cl}_U \Omega = \mathrm{cl}_V f\Omega$. Thus $\partial_V f\Omega \subset f\mathrm{cl}_U \Omega \setminus f\Omega = f\partial_U \Omega$. Hence $f\partial_U \Omega = \partial_V f\Omega$. This proves the surjectivity. Thus $h\colon \partial_U \Omega \to \partial_V f\Omega$ is a continuous closed bijection. Thus h is a homeomorphism. □

Proof of Proposition 4.3.5. Having Lemma 4.3.6 at our disposal, the local degree $\deg(\Omega, f, y)$ is well-defined for each $y \in f\Omega$. Let $h = f|_{\partial_U \Omega}\colon \partial_U \Omega \to$

$\partial_V f\Omega$ and $g = f|_\Omega\colon \Omega \to f\Omega$ be restrictions of f. By Lemma 4.3.6, h is a homeomorphism. Moreover, $\deg(\Omega, f, y) = \deg(\Omega, f, f\Omega) = \deg g$. It suffices to show that $g^*\colon H^n_c(f\Omega) \to H^n_c(\Omega)$ is surjective.

By assumption, $U \setminus A$ is not connected and hence $\mathrm{cl}_E\Omega$ is a proper closed subset of U. Thus, by Corollary 2.5.2, we have that $H^n_c(\mathrm{cl}_U\Omega) = 0$. By the exact sequence of a pair $(\mathrm{cl}_U\Omega, \partial\Omega)$ and Theorem 1.9.2, the diagram

$$\begin{array}{ccccc} H^{n-1}_c(\partial_U\Omega) & \xrightarrow{\partial} & H^n_c(\Omega) & \longrightarrow & 0 \\ \cong\uparrow h^* & & \uparrow g^* & & \\ H^{n-1}_c(\partial_U f\Omega) & \xrightarrow{\partial} & H^n_c(f\Omega) & & \end{array}$$

commutes. Thus the connecting homomorphism $\partial\colon H^{n-1}_c(\partial_U\Omega) \to H^n_c(\Omega)$ is surjective. Since h^* is an isomorphism, we conclude that $g^* \circ \partial$, and hence also g^*, is surjective. □

Proof of Väisälä's theorem. It suffices to consider the case that U is a pre-compact domain in $\mathbb{R}^n$. Let $f\colon U \to V$ be a discrete and open map. Since U is pre-compact, each pre-image $f^{-1}(y)$ is a finite set for $y \in V$. By Theorem 4.2.1, B_f has no interior points.

We show that B_f does not separate U locally at any point. Suppose towards contradiction that B_f separates U locally at some point and let $S \subset B_f$ be the subset of B_f containing all such points. In particularly, $S \neq \emptyset$.

The main part of the proof is to show that there exists a domain $D \subset U$ with the following properties:

1. $D \cap \mathrm{cl}_U S \neq \emptyset$ and $f|_{D\cap \mathrm{cl}_U S}$ is injective,
2. $f|_D\colon D \to fD$ is closed,
3. for each component $W \subset D \setminus \mathrm{cl}_U S$, the restriction $f|_{\mathrm{cl}_D W}\colon \mathrm{cl}_D W \to f(\mathrm{cl}_D W)$ is a homeomorphism.

Suppose, for a moment, that we have found such a domain D. We complete the proof as follows.

Since $\mathrm{cl}_U S \cap D \neq \emptyset$, there exists $x \in S \cap D \subset B_f \cap D$. Since $f|_D\colon D \to fD$ is closed and surjective but not a local homeomorphism at x, we conclude that $f|_D$ is not locally injective at x. Thus there exists two points x_1 and x_2 in D for which $f(x_1) = f(x_2)$. Since $f|_{D\cap \mathrm{cl}_U W}$ is a homeomorphism for each component $W \subset D \setminus \mathrm{cl}_U S$, there exists two components W_1 and W_2 of $W \subset D \setminus \mathrm{cl}_U S$ for which fW_1 and fW_2 are contained in the same component of $f(W \setminus \mathrm{cl}_U S)$. Since $f|_{W_i}$ is both open and closed by assumptions, we now have that fW_1 and fW_2 are the same component of $f(W \setminus \mathrm{cl}_U S)$. Thus we have a homeomorphism

$$h = (f|_{\mathrm{cl}_D W_2})^{-1} \circ (f|_{\mathrm{cl}_D W_1})\colon \mathrm{cl}_D W_1 \to \mathrm{cl}_D W_2.$$

Since $\partial_D W_1 \subset \mathrm{cl}_D S$ and $f|_{\mathrm{cl}_D S}$ is injective, we have that $h|_{\partial_D W_1} = \mathrm{id}$. By Theorem 4.3.4, W_1 and W_2 are the only components of $D \setminus \mathrm{cl}_U S$ and $D \setminus \mathrm{cl}_U S = W_1 \cup W_2$. Since $fW_1 = fW_2$, $f(D \setminus \mathrm{cl}_U S)$ has exactly one component and $f|_{W_i} \colon W_i \to f(D \setminus \mathrm{cl}_U S)$ is a homeomorphism for $i = 1, 2$.

Since $D = W_1 \cup W_2 \cup \mathrm{cl}_D S$ and $fW_1 = fW_2$, we have

$$fD = fW_1 \cup f(\mathrm{cl}_D S).$$

This is a contradiction, since $f(\mathrm{cl}_D S) \cap fW_1 = \emptyset$ and $f(\mathrm{cl}_D S)$ is not open.

It remains to find such a domain D. For each $i \in \mathbb{N}$, let $E_i = \{x \in \mathrm{cl}_U S \colon \# f^{-1} f(x) \le i\}$. Then $\mathrm{cl}_U S = \bigcup_{i \in \mathbb{N}} E_i$. Thus, by Baire's theorem, there exists smallest integer $i_0 \in \mathbb{N}$ for which E_{i_0} has non-empty interior in $\mathrm{cl}_U S$; note that $i_0 \ge 2$. Let $x_1 \in \mathrm{int} E_{i_0}$ and $f^{-1} f(x_1) = \{x_1, \ldots, x_{i_0}\}$.

Let $V \subset U$ be an open set for which $V \cap \mathrm{cl}_U S = \mathrm{int}_{\mathrm{cl}_U S} E_i$. Let now $V_1, \ldots, V_{i_0} \subset W$ be pair-wise disjoint neighborhoods of $x_1, \ldots, x_{i_0}$ and set $\Omega = V_1 \cap \left(f^{-1} \bigcap_{i=1}^{i_0} fV_i \right)$. Then $\# f^{-1}(f(x)) \cap \Omega = 1$ for each $x \in \Omega \cap \mathrm{cl}_U S$.

Since $x_1 \in \mathrm{cl}_U S$, there exists $x_0 \in \Omega \cap \mathrm{cl}_U S$ and a connected neighborhood $G \subset \Omega$ of x_0 for which $G \setminus \mathrm{cl}_U S$ is not connected. By Lemma 3.3.8, there exists a normal neighborhood $D \subset G$ of x_0. By Lemma 3.3.4, $f|_D \colon D \to fD$ is a closed map. Now $D \setminus \mathrm{cl}_U S$ is not connected. Indeed, if $D \setminus \mathrm{cl}_U S$ is connected then $D \setminus \mathrm{cl}_U S$ is contained in a component of $G \setminus \mathrm{cl}_U S$, but this is a contradiction since G is a neighborhood of x.

By construction, $D \cap \mathrm{cl}_U S \ne \emptyset$ and $f|_D \colon D \to fD$ is closed. Furthermore, since $D \subset \Omega$, we have that $f|_{D \cap \mathrm{cl}_U S}$ is injective. Thus it remains to show that, for each component W of $D \setminus \mathrm{cl}_U S$, the restriction $f|_{\mathrm{cl}_D W} \colon \mathrm{cl}_D W \to f(\mathrm{cl}_D W)$ is a homeomorphism.

Let W be a component of $D \setminus E$. Then, by Proposition 4.3.5, $g = f|_W \colon W \to fW$ is a closed mapping and fW is a component of $fV \setminus f(E \cap V)$. Moreover, $y \mapsto \deg(W, f, y)$ is either constant function 1 or -1 in fW. By Lemma 4.3.3, we now conclude that $f|_W \colon W \to fW$ is a homeomorphism.

Since $f|_{\partial_D W} \colon \partial_D W \to f(\partial_D W)$ is a homeomorphism, we conclude that $f|_{\mathrm{cl}_D W}$ is a homeomorphism.

This concludes the proof of Väisälä's theorem. $\square$