III Collocation methods

We want to solve the Volterra integral equation (VIE)

$$u(t) = g(t) + \int_0^t K(t,s)\, u(s)\, ds \qquad (1)$$

on the interval $[0,T] =: I$. Denote

$I_h := \{ t_i : 0 = t_0 < t_1 < \ldots < t_n = T \}$ the mesh \ grid.

Then the intervals are given by

$$e_i := (t_i, t_{i+1}], \quad h_i := t_{i+1} - t_i \quad (0 \leq i \leq n-1)$$

and $h = \max\limits_{i=0 \ldots n-1} h_i$,

Goal: Find "good" approximation $u_h(t)$ to the solution $u(t)$ of (1) such that:

- $u_h(t)$ is defined for all $t \in I$
- $u_h(t)$ can be easily computed on non-uniform meshes.

→ We approximate $u(t)$ by a piecewise polynomial $u_h(t)$.

For a given mesh I_h and integer $m \geq 0$ we define

$$S_m^h(I_h) := \{ p : p|_{e_i} \in \mathbb{P}_m \ , \ i = 0, \dots n-1 \}$$

That means $p \in S_m^h(I_h)$ is a piecewise polynomial of degree m on each Interval e_i.

Note $\dim(S_m^h(I_h)) = n \cdot (m+1)$,

this corresponds to the amount of unknown coefficients!

$\rightarrow$ $m = 0$: piecewise constants

$m = 1$: piecewise linear

1.) Collocation points and equation

Since $\dim(S_m^h(I_h)) = n(m+1)$.

An element $u_h \in S_m^h(I_h)$ contains $n \cdot (m+1)$ unknown coefficients $\Rightarrow$ We need $n \cdot (m+1)$ points in $[0,T]$ at which $u_h(t)$ must satisfy the VIE.

These points are the collocation points.

Let $0 \le c_0 < \dots < c_m \le 1$, $c_i \in [0,1]$

are called the collocation parameters.

The set

$$X_h := \{ t_i + c_j h_i : j = 0, \dots, m ; \ i = 0, \dots, N-1 \}$$

are called the collocation points. Each interval has $m+1$ points.

<u>Now:</u> Find $u_h \in S_m^h(I_h)$ so that it satisfies the VIE at the points X_h:

$$u_h(t) = g(t) + \int_0^t K(t,s)\, u_h(s)\, ds \quad , t \in X_h .$$

u_h is called the collocation solution.

One can define the iterated solution

$$u_h^{it}(t) := g(t) + \int_0^t K(t,s)\, \underline{u_h}(s)\, ds \ , \quad \underline{t \in [0,T]} .$$

2 Collocation for ordinary differential equations

Define for $m \geq \cancel{0}\, 1$

$$S_m^0(I_h) = \{ v \in C(I) : v|_{e_i} \in \mathbb{P}_m,\ i = 0, \dots, \cancel{N}-1 \}$$

the space of globally continuous piecewise polynomials

$$\dim (S_m^0(I_h)) = \cancel{N}\, n \cdot m + 1$$

(Note reduction of dimension due to continuity)

We seek to compute the collocation solution $u_h \in S_m^0(I_h)$ for the ODE:

(2) $$u'(t) = f(t, u(t)),\ t \in I,\ u(0) = u_0$$

Determined by the col. eq.

$$u_h(t) = f(t, u_h(t)),\ t \in X_h,\ u_h(0) = u_0.$$

X_h are the collocation points. ~~as defined earlier~~,

~~where $|X_h| = n \cdot m$.~~

! Here $X_h := \{ t_i + c_j h_i : 0 \leq c_1 < \dots < c_m \leq 1,\ i = 0, \dots, n-1 \}$

Then $|X_h| = n \cdot m$

Computational representation

Let $L_j(v) := \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{v - c_k}{c_j - c_k}$, $v \in [0,1]$ $(j = 1, \dots, m)$

$(m = 1 \Rightarrow L_1(v) \equiv 1)$

are the Lagrange polynomials for the collocation ~~points~~ parameters

set $Y_{i,j} := u_h'(t_i + c_j h_i)$ and

$$u_h'(t_i + v h_i) = \sum_{j=1}^{m} L_j(v)\, Y_{i,j} \quad , v \in (0,1]$$

the local representation of $u_h \in S_m^0(I_h)$ on e_i:

(3) $$u_h(t_i + v h_i) = u_h(t_i) + h_i \sum_{j=1}^{m} \beta_j(v)\, Y_{i,j} \quad , v \in [0,1]$$

with $\beta_j(v) := \int_0^v L_j(s)\, ds$.

$\rightarrow$ Compute $\{Y_{i,j}\}$, $i = 0, \dots, n-1$:

(4) $$Y_{i,k} = f\left(t_i + c_k h_i,\; y_i + h_i \sum_{j=1}^{m} a_{k,j} Y_{i,j}\right)$$

$k = 1 \dots m$

$y_i = u_h(t_i)$ and $a_{k,j} := \beta_j(c_k)$

(4) is called the collocation equation

for $t = t_i + c_k h_i$

Together (3) and (4) represent a so-called "m-stage continuous implicit Runge-Kutta method" for solving the ODE:

$$u'(t) = f(t, u(t)) \ , \quad t \in [0,T] \ , \ u(0) = u_0 .$$

Example → p. 10

<u>Convergence</u> For arbitrary $\{c_k\}$ (and $u \in C^d(I)$, with $d \geq m+1$):

$$\| u^{(j)} - u_h^{(j)} \|_\infty \leq C h^m \quad (j = 0,1)$$

[Higher order can be achieved for Gauss-Legendre points; $O(h^{2m})$ for u_h and $O(h^m)$ for u_h'.]

3 Collocation for VIE

We have

$$u(t) = g(t) + \int_0^t K(t,s)\, u(s)\, ds \ , \ t \in [0,T]$$

Collocation equation: Find $u_h \in S_m^h(I_h)$ such that

$$(5) \qquad u_h(t) = g(t) + \int_0^t K(t,s)\, u_h(s)\, ds \ , \quad t \in X_h$$

Following the last section: ~~we will have~~

For $t = t_i + v h_i \in e_i$ $(v \in (0,1])$ we have

(6) $u_h(t) = \sum_{j=0}^{m} L_j(v) U_{i,j}$, $v \in (0,1]$

and $U_{i,j} := u_h(t_i + c_j h_i)$ unknown.

How to compute $u_h(t)$?

Insert (6) into (5) with $t = t_i + c_j h_i$

$\Rightarrow$ System of equations: $U_i := (U_{i,0}, U_{i,1}, \dots, U_{i,m})^T$:

(7) $$\left[I_m - h_i A_i \right] U_i = g_i + \sum_{k=0}^{i-1} h_k A_{i,k} U_k$$

$A_i \in \mathbb{R}^{(m+1)\times(m+1)}$, I_m identity matrix.

$\Rightarrow$ For each $i = 0, \dots, n-1$ solve the linear system (7) for $U_i = (U_{i,0}, \dots U_{i,m})$ where $U_{i,j} = u_h(t_i + c_j h_i)$ and $g_{i,j} := \{ g(t_i + c_j h_i) \; j = 0, \dots, m$.

The matrices ~~can be compute~~ are given by

$$A_i := \left(\int_0^{c_j} K(t_i + c_j h_i, t_i + s h_i) L_k(s) \, ds \right)$$

$j, k = 0, \dots m$

and for ~~ ~~ $s = 0, \dots, i-1$

$$A_{i,s} := \left(\int_0^1 K(t_i + c_j h_i, t_s + \tau h_s) L_k(\tau)\, d\tau \right)_{j,k=0,\dots,m}$$

If we know the solution of (7) then the collocation solution on $e_i = [t_i, t_{i+1}]$ is given by $u_h(t_i + v h_i) = \sum_{j=0}^{m} L_j(v) U_{i,j} \quad v \in (0,1]$

→ We need to compute A_i and $A_{i,s}$ by some quadrature formula. → Exercise

Theorem ~~ ~~ 1 (Convergence)

(a) If $u \in C^d(I)$ $(d \geq m+1)$ then for general $\{c_j\}$

$$\|u - u_h\|_\infty \leq C h^{m+1}$$

(b) If $u \in C^d(I)$ $(d \geq m+2)$ and the $\{c_j\}$ are such that ~~ ~~ $\int_0^1 \prod_{j=0}^{m} (s - c_j)\, ds = 0$ (Gauss points)

then $\|u - u_h^{it}\|_\infty \leq C h^{m+2}$

Example ~~a~~ for ODEs

Let $u_h \in S_1^{0}(I_h)$, $(m=1)$ and $c_1 = \theta \in [0,1]$

$L_1(v) \equiv 1 \quad \Rightarrow \quad \beta_1(v) = v, \; a_{11} = \theta$

Then (3) is

$$u_h(t_i + v h_i) = y_i + h_i v Y_{i,1} \quad \text{and}$$

$$(4) \rightarrow \quad Y_{i,1} = f(t_i + \theta h_i, y_i + h_i \theta Y_{i,1})$$

~~By setting $v=1$ we can combine this to~~

This can be rewritten to

$$u_h(t_i + v h_i) = (1-v) y_i + v y_{i+1} \qquad v \in [0,1]$$

$$y_{i+1} = y_i + h_i f(t_i + \theta h_i, (1-\theta) y_i + \theta y_{i+1})$$

$\Rightarrow \quad \theta = 1$ implicit Euler

$\theta = \frac{1}{2}$ implicit midpoint rule

$\theta = 0$ explicit Euler.