Computational methods for Integral Equations — Exercise 2 (March 13, 2017)

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1. *First order Runge-Kutta methods*
   We aim to implement the two different Euler methods for solving an ordinary differential equation. See the example in the Lecture: lecture 2, page 10.

   Consider the following differential equation on $I = [0, 1]$

   $$u' = 2ut + t^3, \; u(0) = 1$$

   The solution is given by $u(t) = \frac{3}{2}e^{t^2} - \frac{1}{2}(t^2 + 1)$.

   i.) Start with the explicit Euler method, i.e. $\theta = 0$. Note that $y_i = u_h(t_i)$ and hence it follows $y_0 = u_h(t_0) = u_0$. You can then easily compute the following iterates as

   $$y_{i+1} = y_i + h_i f(t_i, y_i).$$

   ii.) Compare the results to the implicit Euler, i.e. $\theta = 0$. For solving the implicit equation

   $$y_{i+1} = y_i + h_i f(t_i + h_i, y_{i+1}),$$

   you can use Newton's method (see for instance on Wikipedia under Newton's method) for the function

   $$\phi(y) = y - (y_i + h_i f(t_i + h_i, y)).$$

   Note that the derivative ϕ' is with respect to y and y_i is treated as a constant.

2. *Convergence of the Euler methods*
   What can you say about the convergence and error of the two methods implemented in Exercise 1?

3. *Solving a Volterra Integral Equation*
   We want to solve the Volterra integral equation

   $$u(t) = t - \int_0^t (t - s)u(s)ds,$$

   with the solution $u(t) = \sin(t)$. The goal is to implement the collocation method for $m = 0$, $c_0 = 1$ and $I = [0, \pi]$. Note that in this setting we have

   $$(1 - h_i a_i)u_i = g_i + \sum_{k=0}^{i-1} h_k a_{i,k} u_k,$$

where the matrices reduce to scalars and $u_0 = g(0)$. For computing the integrals in a_i and $a_{i,k}$ you can use a quadrature rule of your choice.

4. *Convergence of the collocation method*
   What can you say about the convergence of your algorithm in Exercise 3?