LECTURE NOTES ON MULTILINEAR WEIGHTED INEQUALITIES AND SHARP BOUNDS

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These lecture notes are mainly based on the articles [37] by A.K. Lerner, S. Ombrosi, C. Pérez, R.H. Torres and R. Trujillo-González, [15] by W. Damián, A.K. Lerner and C. Pérez, [9] by W. Chen and W. Damián, [38] by K. Li, K. Moen and W. Sun and [14] by W. Damián, M. Hormozi and K. Li.

Prerequisites

We will assume a prior knowledge of real and functional analysis, measure and integration as well as some basic inequalities such us Hölder and Minkowski. Previous knowledge of the unweighted multilinear Calderón–Zygmund theory contained in [21] will be desirable but not necessary to understand the course.

Purpose and description

The purpose of these notes is to give a short but detailed introduction to multilinear weighted inequalities and the usual techniques of proof in the area.

On one hand, we start describing the main object in this area, the multilinear maximal function, and how it controls the class of multilinear Calderón–Zygmund operators and allow us to define the right class of multiple weights. We also prove the generalization of Muckenhoupt's one and two-weight problems for the multilinear maximal function $\mathcal{M}$ as well as some multiple (sharp) weighted inequalities for multilinear maximal functions and sparse operators.

On the other hand, we give a pointwise control of multilinear Calderón–Zygmund operators of Dini type by sparse operators. As a consequence of this result and using some mixed weighted bounds for a general class of sparse operators, we will be able to show similar bounds for several multilinear operators such us Calderón–Zygmund operators, their commutators with BMO functions, square functions and Fourier multipliers.

1. Introduction

The origin of the modern theory of weighted inequalities can be traced back to the works of R. Hunt, B. Muckenhoupt, R. Wheeden, R. Coifman, and C. Fefferman in the decade of the 70's. The basic problem concerning weighted inequalities consists in determining under which conditions a given operator, initially bounded on $L^p(\mathbb{R}^n)$, is bounded on

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$L^p(\mathbb{R}^n, \mu)$, where μ is an absolutely continuous measure with respect to Lebesgue measure, i.e. $d\mu = wdx$. Here, w denotes a non-negative locally integrable function on $\mathbb{R}^n$ that is positive almost everywhere, that is called a *weight*.
A sustained research period was started with the groundbreaking work of Muckenhoupt [45]. In this work he characterized the class of weights u, v for which the following weak inequality for the Hardy–Littlewood maximal operator and for $1 \leq p < \infty$ holds

$$\sup_{\lambda>0} \lambda^p \int_{\{Mf>\lambda\}} u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad f \in L^p(v). \tag{1.1}$$

$$||M(f)||_{L^{p,\infty}(u)} \leq C||f||_{L^p(v)}. \tag{1.2}$$

This condition on the weights is known as A_p condition, namely

$$[u, v]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\right)^{p-1} < \infty, \quad p > 1,$$

where the supremum is taken over all the cubes in $\mathbb{R}^n$. Note that when $p = 1$, the term $(⨍_Q v(x)^{-\frac{1}{p-1}})^{p-1}$ must be understood as $(\operatorname{ess\,inf}_Q v)^{-1}$. Although weights in the A_p class are also known as Muckenhoupt weights, it is worth mentioning that variant of this condition was previously introduced by Rosenblum in [50]. In the particular case $u = v$ and $p > 1$, Muckenhoupt also proved that the following strong estimate

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad f \in L^p(v),$$

holds if and only if v satisfies the A_p condition.
From that point on, the interest of harmonic analysts focused on studying weighted inequalities for the classical operators such as the Hilbert and Riesz transforms and other singular integral operators leading to a wide literature on one-weight norm inequalities.
However, the problem of finding a condition on the weights u, v satisfying the strong estimate above was much more complicated. It was not until 1982 that E. Sawyer [51] characterized the two weight inequality, showing that $M : L^p(v) \longrightarrow L^p(u)$ if and only if the pair of weights (u, v) satisfies the following testing condition known as Sawyer's S_p condition

$$[u, v]_{S_p} = \sup_Q \left(\frac{\int_Q M(\chi_Q\sigma)^p u dx}{\sigma(Q)}\right)^{1/p} < \infty, \tag{1.3}$$

where $\sigma = v^{1-p'}$ and $1 < p < \infty$. Observe that condition (1.3) involves the operator under study itself and, for this reason, it is difficult either to check or use it to construct examples of weights for applications. This difficulty together with the fact that these conditions are just defined for particular operators motivated the development of different sufficient conditions, close in form to the A_p condition.
The classical results mentioned so far did not reflect the quantitative dependence of the $L^p(w)$ operator norm in terms of the relevant constant involving the weights since they were qualitative properties. Therefore, the relevant question then was to determine the precise sharp bounds of a given operator in $L^p(w)$, whenever $w \in A_p$.

The first author who studied this problem for the Hardy–Littlewood maximal operator was S. Buckley, a Ph.D. student of R. Fefferman, who proved in [5],

$$||M||_{L^p(w)} \leq C\, p'\, [w]_{A_p}^{\frac{1}{p-1}}, \tag{1.4}$$

where C is a dimensional constant. We say that the above inequality is sharp in the sense that we cannot replace the exponent on the weight constant by an smaller one. Buckley also proved another quantitative result related to the weak estimate for the Hardy–Littlewood maximal operator as an application of the classical covering lemmas. More precisely,

$$||M||_{L^p(w)\to L^{p,\infty}(w)} \leq C[w]_{A_p}^{1/p}, \tag{1.5}$$

where C is a dimensional constant. In fact, it can be easily proved that the operator norm and the weight constant in (1.5) are comparable, whereas in (1.4) this result is false (see [25] for further details).
Following the spirit of Buckley's results, a similar problem was studied by J. Wittwer, another Ph.D. student of R. Fefferman, for the martingale operator and the square function in [54] and [55], respectively. Later on, regarding the two-weight problem for the Hardy–Littlewood maximal function, K. Moen found in [43] a quantitative form of E. Sawyer's result in terms of Sawyer's S_p condition (1.3). Namely

$$||M||_{L^p(v)\longrightarrow L^p(u)} \approx [u,v]_{S_p}. \tag{1.6}$$

Although maximal functions are relevant operators in harmonic analysis, singular integrals are probably the central operators in this field. The term singular integral refers to a wide class of operators that are (formally) defined, as integral operators in the following way

$$Tf(x) = \int K(x,y)f(y)dy,$$

where K is a singular kernel in the sense that it is not locally integrable.
The prototype or most representative example of this class of operators is the Hilbert transform in the real line, namely

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

In the light of the previous results, the relevant problem then was trying to determine the sharp constant in the corresponding weighted inequality for Calderón–Zygmund singular integral operators. Concerning this problem, the next relevant step in this direction was given by K. Astala, T. Iwaniec and E. Saksman in [3]. They studied the Beurling transform (also known as the Ahlfors-Beurling transform) defined as follows

$$Bf(z) = p.v. \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} dw.$$

This Calderón–Zygmund operator is one of the most important singular integral operators related to complex variables, quasi-conformal mappings and the regularity theory of the Beltrami equation. In fact, in [3] the authors were interested in finding the smallest $q < 2$ such that the solutions of the Beltrami equation

$$\bar{\partial} f = \mu \partial f$$

that belong to the Sobolev space $W^{1,q}_{loc}$ also belong to the better space $W^{1,2}_{loc}$ (i.e. the solutions are quasi-regular). Here μ is a bounded function such that $||\mu||_\infty = k < 1$. Lately, K. Astala [2] proved that $q > k+1$ is sufficient. On the other hand, T. Iwaniec and G.J. Martin [27] found examples showing that, in general, the result does not hold for $q < k+1$.

In [3] the authors also pointed out that in the case $q = k+1$, the quasi-regularity would be a consequence of a linear bound of $||B||_{L^p(w)}$ for $p \geq 2$ in terms of the weight constant. In fact, they conjectured the following bound for the Beurling operator

$$||B||_{L^p(w)} \leq c_p [w]_{A_p}, \qquad p \geq 2, \tag{1.7}$$

which was proved by S. Petermichl and A. Volberg in [49]. This conjecture revealed the importance of finding a bound on the norm of a given operator in terms of the weight constant. Another feature of the theory is the relevance of the case $p = 2$. It is due to the fact that, as a consequence of Rubio de Francia's extrapolation theorem obtained in [16], it suffices to obtain a linear bound in the case $p = 2$ since it is the starting point to derive sharp bounds for all p. We refer the interested reader to [13] for a simpler proof of the precise extrapolation theorem, which was inspired by the work of Duoandikoetxea [17].

The next important advance in this area was due to S. Petermichl [47] who proved the optimal bounds for the Hilbert transform. Shortly after, she extended this result to the Riesz transforms in [48]. Lately, O. Beznosova proved the analogous linear bound for discrete paraproduct operators in [4].

It was then that the so-called A_2 conjecture gathered more importance. This conjecture claimed that the dependence for a Calderón–Zygmund operator will be linear on the A_2 constant, namely

$$||T||_{L^2(w)} \leq C [w]_{A_2}. \tag{1.8}$$

As mentioned before, from (1.8) it is possible to extrapolate to get the A_p dependence. More precisely,

$$||T||_{L^p(w)} \leq C [w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)}, \tag{1.9}$$

where the dimensional constant C depends also on p and T.

In 2010, the sharp A_2 bound for a large family of Haar shift operators that included dyadic operators was obtained by M. Lacey, S. Petermichl and M.C. Reguera in [31]. After that, D. Cruz-Uribe, J.M. Martell and C. Pérez proved a more flexible result in [13] that could be applied to many different operators and whose proof avoids Bellman functions as well as two-weight norm inequalities.

After many intermediate results by others, the A_2 conjecture was solved in full generality by T. Hytönen in [24] using a very different and interesting probabilistic approach. Shortly after, A.K. Lerner gave a simpler and beautiful proof in [35] based on the use of dyadic sparse operators and the so-called local mean oscillation formula. Lately, K. Moen [44] derived sharp weighted bounds for sparse operators for all p, $1 < p < \infty$, avoiding the use of extrapolation.

After the solution of the A_2 conjecture, several improvements of this and other results were obtained in [25] by T. Hytönen and C. Pérez. The underlying idea of this work was to replace a portion of the A_2 constant by another smaller constant defined in terms of the A_∞ constant given by

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q). \tag{1.10}$$

This functional was implicitly considered by N. Fujii in [18] to provide a characterization of the A_∞ class of weights and later it was rediscovered by M. Wilson in [53]. It is smaller than the more classical A_∞ condition due to Hrusčěv

$$[w]^H_{A_\infty} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right) \exp\left(\frac{1}{|Q|}\int_Q \log w^{-1}\right),$$

as it was shown in [25] for the particular case of weights of the form $w = t\chi_E + \chi_{\mathbb{R}\setminus E}$ with $t \geq 3$.

On the one hand, in [25] an improvement of Buckley's estimate for the Hardy–Littlewood maximal function is proved. Namely, for $p > 1$,

$$||M||_{L^p(w)} \leq Cp'([w]_{A_p}[\sigma]_{A_\infty})^{1/p}, \tag{1.11}$$

where C is a dimensional constant and $\sigma = w^{1-p'}$. This result improves significantly Buckley's bound since

$$([w]_{A_p}[w]_{A_\infty})^{1/p} \lesssim ([w]_{A_p}[w]_{A_p}^{\frac{1}{p-1}})^{1/p} \lesssim [w]_{A_p}^{\frac{1}{p-1}}.$$

On the other hand, in [25] the A_2 theorem (as well as its L^p counterpart) was improved obtaining the following mixed sharp $A_2 - A_\infty$ estimate for singular integral operators

$$||T||_{L^2(w)} \leq C[w]_{A_2}^{1/2}([w^{-1}]_{A_\infty} + [w]_{A_\infty})^{1/2}, \tag{1.12}$$

which is the starting point for proving analogous sharp bounds for other operators such as commutators and their iterates as well.

2. Preliminaries on multilinear Calderón–Zygmund theory

The multilinear Calderón–Zygmund theory can be traced back to the works of R. Coifman and Y. Meyer [11] in the seventies. Their work was oriented towards the study of certain singular integral operators, such us the commutator of Calderón. This theory, far from being a mere generalization of the linear theory, appears naturally in harmonic analysis. The boundedness results for the bilinear Hilbert transform obtained by M. Lacey and C. Thiele [32, 33], motivated the development of a systematic treatment of general multilinear Calderón–Zygmund operators. In this respect, the work of L. Grafakos and R. Torres [21] set the bases of the unweighted multilinear Calderón–Zygmund theory.

Here, we introduce the notion of Calderón–Zygmund operator in the multilinear scenario as well as some (unweighted) boundedness properties that may be found in [21].

Definition 2.1. Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n).$$

We say that T is an m-linear Calderón-Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m, \tag{2.1}$$

for all $x \notin \cap_{j=1}^m \operatorname{supp} f_j$,

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{A}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l|\Big)^{mn}}, \tag{2.2}$$

and

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leq \frac{A|y_j - y_j'|^{\epsilon}}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l|\Big)^{mn+\epsilon}}, \tag{2.3}$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y_j'| \leq \dfrac{1}{2} \max\limits_{0 \leq k \leq m} |y_j - y_k|$.

Some basic boundedness properties of multilinear Calderón–Zygmund operators are stated in the following theorem.

Theorem 2.2. *Let T be a multilinear Calderón–Zygmund operator. Let p, p_j numbers satisfying $\frac{1}{m} \leq p < \infty$, $1 \leq p_j \leq \infty$, and $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = \frac{1}{p}$. Then, all the statements below are valid:*

- **(i):** *When all $p_j > 1$, then T can be extended to a bounded operator from $L^{p_1} \times \ldots \times L^{p_m}$ into L^p, where L^{p_k} should be replaced by L_c^∞ if some $p_k = \infty$.*
- **(ii):** *When some $p_j = 1$, then T can be extended to a bounded operator from $L^{p_1} \times \ldots \times L^{p_m}$ into $L^{p,\infty}$, where again L^{p_k} should be replaced by L_c^∞ if some $p_k = \infty$.*
- **(iii):** *When all $p_j = \infty$, then T can be extended to a bounded operator from the m-fold product $L_c^\infty \times \ldots \times L_c^\infty$ into BMO.*

Observe that when all the indexes $p_j = 1$, it is obtained the generalization to the multilinear setting of the weak type $(1, 1)$ boundedness for classical singular integral operators. Namely, the corresponding endpoint space to bound singular integral operators in the multilinear setting is now the m-fold product $L^1 \times \ldots \times L^1$ and, by homogeneity, it is mapped into $L^{1/m,\infty}$, i.e.,

$$T : L^1(\mathbb{R}^n) \times \ldots \times L^1(\mathbb{R}^n) \longrightarrow L^{1/m,\infty}(\mathbb{R}^n). \tag{2.4}$$

In Section 10, we will introduce a more general class of Calderón–Zygmund operators which verifies weaker regularity conditions on the kernel. We extend the previous boundedness results to this wider class of operators as well as for their maximal truncation operator. Those results can be found in Appendix A.

3. The multilinear maximal function

The question of the existence of an appropriate multilinear maximal function and a multiple weight theory was posed in [22]. Although the class of Calderón–Zygmund operators was controlled by $\prod_{j=1}^m Mf_j$, as shown in [46], it was not clear whether this control was optimal and whether the conditions on the weights w_j for which

$$T : L^{p_1}(w_1) \times \ldots \times L^{p_m}(w_m)$$

holds could be improved. In [37], it was introduced a multilinear maximal operator strictly smaller than the m-fold product of M, which gives the right classes of multiple weights for m-linear Calderón–Zygmund operators. In this section, we introduce this operator and a pointwise control of Calderón–Zygmund operators which improves that in [46].
Given $\vec{f} = (f_1, \ldots, f_m)$, we define the multi(sub)linear maximal operator $\mathcal{M}$ by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where the supremum is taken over all cubes containing x. With some abuse of the language, we will refer to $\mathcal{M}$ as the multilinear maximal function, even though it is obvious that it is only sublinear in each entry.
Since this operator is smaller than the m-fold product of Hardy–Littlewood maximal functions, as a consequence of Hölder's inequality and the corresponding version for weak spaces (see [20, p. 15]), it satisfies the corresponding natural unweighted estimates. Namely,

$$\begin{aligned} &\mathcal{M} : L^1 \times \overset{m}{\cdots} \times L^1 \to L^{1/m,\infty}, \\ &\mathcal{M} : L^{p_1} \times \overset{m}{\cdots} \times L^{p_m} \to L^p, \end{aligned} \tag{3.1}$$

where $1 < p_1, \ldots, p_m < \infty$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$.
The importance of this operator stems from the fact that it controls the class of multilinear Calderón–Zygmund operators. The following result, which can be found in [37], was originally proved by J. Álvarez and C. Pérez in the linear setting in [1].

Theorem 3.1. *Let T be an m-Calderón–Zygmund operator and let $\delta > 0$ such that $\delta < 1/m$. Then for all $\vec{f}$ in any product of $L^{q_j}(\mathbb{R}^n)$, with $1 \le q_j < \infty$,*

$$M_\delta^\sharp(T(\vec{f}))(x) \lesssim \mathcal{M}(\vec{f})(x). \tag{3.2}$$

Proof. Fix $x \in \mathbb{R}^n$ and a cube Q containing x. To prove (3.2)it suffices to prove that for any $0 < \delta < 1/m$

$$\left(\frac{1}{|Q|} \int_Q \left| |T(\vec{f})(z)|^\delta - |c_Q|^\delta \right| dz \right)^{1/\delta} \lesssim \mathcal{M}(\vec{f})(x), \tag{3.3}$$

for a certain constant c_Q to be determined later on. Having into account that $||\alpha|^r - |\beta|^r| \le |\alpha - \beta|^r$, $0 < r < 1$, we only need to show

$$\left(\frac{1}{|Q|} \int_Q |T(\vec{f})(z) - c_Q|^\delta dz \right)^{1/\delta} \lesssim \mathcal{M}(\vec{f})(x). \tag{3.4}$$

Let $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j\chi_{Q^*}, j = 1,\dots,m$ and $Q^* = 3Q$. Then,

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \big(f_j^0(y_j) + f_j^\infty(y_j)\big)\\ &= \sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\dots f_m^{\alpha_m}(y_m)\\ &= \prod_{j=1}^m f_j^0 + {\sum}' f_1^{\alpha_1}(y_1)\dots f_m^{\alpha_m}(y_m),\end{aligned}$$

where each term of $\sum'$ contains at least one $\alpha_j \neq 0$. We can write then

$$T(\vec{f})(z) = T(\vec{f^0})(z) + {\sum}' T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(z). \tag{3.5}$$

Applying Kolmogorov's inequality to the term

$$T(\vec{f^0}(z)) = T(f_1^0,\dots,f_m^0)(z)$$

with $p=\delta$ y $q = 1/m$, it follows that

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |T(\vec{f^0})(z)|^\delta dz\right)^{1/\delta} &\lesssim_{m,\delta} \|T(\vec{f^0})(z)\|_{L^{1/m,\infty}(Q,\frac{dx}{|Q|})}\\ &\lesssim \prod_{j=1}^m \frac{1}{|3Q|}\int_{3Q}|f_j(y_j)|\,dy_j\\ &\lesssim \mathcal{M}(\vec{f})(x),\end{aligned}$$

since $T : L^1\times\dots\times L^1 \to L^{1/m,\infty}$.
In order to estimate the other terms in (3.5), we set now

$$c = {\sum}' T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x),$$

and we will show that, for any $z\in Q$, we also get an estimate of the form

$$ {\sum}' |T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(z) - T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)| \lesssim \mathcal{M}(\vec{f})(x). \tag{3.6}$$

Consider first the case when $\alpha_1 = \ldots = \alpha_m = \infty$ and define

$$T(\vec{f^\infty}) = T(f_1^\infty,\dots,f_m^\infty).$$

Using the regularity of the kernel of T, for any $z\in Q$, we obtain

$$\begin{aligned}&|T(\vec{f^\infty})(z) - T(\vec{f^\infty})(x)|\\ &\lesssim \int_{(\mathbb{R}^n\setminus 3Q)^m} \frac{|x-z|^\varepsilon}{(|z-y_1|+\dots+|z-y_m|)^{nm+\varepsilon}}\prod_{i=1}^m |f_i(y_i)|d\vec{y}\\ &\lesssim \sum_{k=1}^\infty \int_{(3^{k+1}Q)^m\setminus(3^kQ)^m} \frac{|x-z|^\varepsilon}{(|z-y_1|+\dots+|z-y_m|)^{nm+\varepsilon}}\prod_{i=1}^m |f_i(y_i)|d\vec{y}\end{aligned}$$

$$\lesssim \sum_{k=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(3^k|Q|^{1/n})^{nm+\varepsilon}} \int_{(3^{k+1}Q)^m} \prod_{i=1}^{m} |f_i(y_i)| d\vec{y}$$
$$\lesssim \sum_{k=1}^{\infty} \frac{1}{3^{k\varepsilon}} \prod_{i=1}^{m} \langle |f_i| \rangle_{3^{k+1}Q} \lesssim \mathcal{M}(\vec{f})(x),$$

where $E^m = E \times \cdots \times E$ and $d\vec{y} = dy_1 \dots dy_m$.
What remains to be considered are the terms in (3.6) such that $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for some $\{j_1, \dots, j_l\} \subset \{1, \dots, m\}$ and $1 \le l < m$. Using again the regularity of the kernel,

$$|T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x)|$$
$$\lesssim \prod_{j \in \{j_1,\dots,j_l\}} \int_{3Q} |f_j| dy_j \int\limits_{(\mathbb{R}^n \setminus 3Q)^{m-l}} \frac{|x-z|^{\varepsilon} \prod_{j \notin \{j_1,\dots,j_l\}} |f_j| dy_j}{(|z-y_1| + \cdots + |z-y_m|)^{nm+\varepsilon}}$$
$$\lesssim \prod_{j \in \{j_1,\dots,j_l\}} \int_{3Q} |f_j| dy_j \sum_{k=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(3^k|Q|^{1/n})^{nm+\varepsilon}} \int\limits_{(3^{k+1}Q)^{m-l}} \prod_{j \notin \{j_1,\dots,j_l\}} |f_j| dy_j$$
$$\lesssim \sum_{k=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(3^k|Q|^{1/n})^{nm+\varepsilon}} \int_{(3^{k+1}Q)^m} \prod_{i=1}^{m} |f_i(y_i)| d\vec{y},$$

and we arrive at the expression considered in the previous case. This gives (3.6) and concludes the proof of the theorem.

□

4. Weak type estimate for $\mathcal{M}$

The previous pointwise control of multilinear Calderón–Zygmund operators by $\mathcal{M}$ opened up the possibility of considering more general weights. In [37], the authors exploited this possibility proving a natural extension to the multilinear setting of Muckenhoupt's two-weight theorem.

Theorem 4.1. *Let $1 \le p_j < \infty, j = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let ν and w_j be weights. Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{4.1}$$

holds for any $\vec{f}$ if and only if

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q \nu \Big)^{1/p} \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'} \Big)^{1/p_j'} < \infty, \tag{4.2}$$

where $\Big(\frac{1}{|Q|} \int_Q w_j^{1-p_j'} \Big)^{1/p_j'}$ in the case $p_j = 1$ must be understood as $(\operatorname{ess\,inf}_Q w_j)^{-1}$.

Proof of Theorem 4.1. The proof is very similar to that in the linear situation (see, for instance, [19, 20]). Let us consider first the case when $p_j > 1$ for all $j = 1, \dots, m$. Assume that $\mathcal{M}$ satisfies (4.1)), namely,

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{4.3}$$

then we can write for every $\vec{f} = (f_1, \ldots, f_m)$,

$$\nu\left(\{x \in \mathbb{R}^n : \mathcal{M}(\vec{f})(x) > t\}\right)^{\frac{1}{p}} \leq \frac{C}{t} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} |f_j(y_j)|^{p_j} w_j\right)^{\frac{1}{p_j}}, \tag{4.4}$$

where p is given as in the assumptions and $t > 0$. Suppose without loss of generality that $\vec{f} \geq 0$, i.e. $f_j \geq 0$, $j = 1, \ldots, m$. Since $\mathcal{M}(\vec{f})(x) \geq \prod_{j=1}^{m} \langle |f_j| \rangle_Q$ for all $x \in Q$, it follows from (4.4) that for all $t < \prod_{j=1}^{m} \langle |f_j| \rangle_Q$, we have that

$$\nu(Q)^{\frac{1}{p}} \leq \nu(\{x \in \mathbb{R}^n : \mathcal{M}(f_1, \ldots, f_m)(x) > t\})^{1/p} \leq C t^{-1} \prod_{j=1}^{m} ||f_j||_{L^{p_j}(w_j)}. \tag{4.5}$$

Taking $f_j 1_Q$ instead of f_j, $j = 1, \ldots, m$, in (4.5), we deduce that

$$\nu(Q)^{\frac{1}{p}} \prod_{j=1}^{m} \langle f_j \rangle_Q \leq C \prod_{j=1}^{m} ||f_j \chi_Q||_{L^{p_j}(w_j)}. \tag{4.6}$$

Next, taking $f_j = w_j^{1-p_j'}$, we obtain

$$\left(\int_Q \nu\right)^{\frac{1}{p}} \left(\prod_{j=1}^{m} \frac{1}{|Q|} \int_Q w_j^{1-p_j'}\right) \left(\prod_{j=1}^{m} \int_Q w_j^{(1-p_j')p_j} w_j\right)^{-\frac{1}{p_j}} < C.$$

Note that

$$\begin{aligned} &\left(\prod_{j=1}^{m} \frac{1}{|Q|} \int_Q w_j^{1-p_j'}\right) \left(\prod_{j=1}^{m} \int_Q w_j^{(1-p_j')p_j} w_j\right)^{-\frac{1}{p_j}} \\ &= \frac{1}{|Q|^m} \prod_{j=1}^{m} \left(\int_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}}. \end{aligned}$$

Then, we have that

$$\left(\fint_Q \nu\right)^{\frac{1}{p}} \prod_{j=1}^{m} \left(\fint_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}} < C, \tag{4.7}$$

for every cube Q, showing (4.2). To prove the converse, assume that (4.2) holds. Using Hölder's inequality, we obtain

$$\begin{aligned}\left(\fint_Q \nu\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\fint_Q |f_j|\right) &\leq \left(\fint_Q \nu\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\fint_Q w_j^{-\frac{p_j'}{p_j}}\right)^{\frac{1}{p_j'}} \left(\fint_Q |f_j|^{p_j} w_j\right)^{\frac{1}{p_j}} \\ &\leq C \prod_{j=1}^m \left(\fint_Q |f_j|^{p_j} w_j\right)^{\frac{1}{p_j}}. \end{aligned} \tag{4.8}$$

The previous inequality applied to cubes $Q(x,r)$ centred in x with radious $r>0$, yields to

$$\prod_{j=1}^m \langle |f_j| \rangle_{Q(x,r)} \leq \frac{C}{\nu(Q(x,r))^{\frac{1}{p}}} \prod_{j=1}^m ||f_j \chi_{Q(x,r)}||_{L^{p_j}(w_j)}.$$

Therefore,

$$\begin{aligned}\mathcal{M}(\vec{f})(x) &\leq C \prod_{j=1}^m \left(\frac{1}{\nu(Q(x,r))} \int_{Q(x,r)} |f_j|^{p_j} w_j \frac{\nu}{\nu} dy_j\right)^{\frac{1}{p_j}} \\ &\leq C \prod_{j=1}^m M_\nu^c \left(|f_j|^{p_j} \frac{w_j}{\nu}\right)(x)^{\frac{1}{p_j}},\end{aligned}$$

where M_ν^c denotes the weighted centred maximal function. Now, using the fact that M_ν^c is weak $(1,1)$ with respect to the weight ν and using the Hölder's inequality for weak spaces, it follows that

$$\begin{aligned}\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} &\leq C\|\prod_{j=1}^m M_\nu^c(|f_j|^{p_j} w_j/\nu)^{1/p_j}\|_{L^{p,\infty}(\nu)} \\ &\leq C \prod_{j=1}^m \|M_\nu^c(|f_j|^{p_j} w_j/\nu)^{1/p_j}\|_{L^{p_j,\infty}(\nu)} \\ &= C \prod_{j=1}^m \|M_\nu^c(|f_j|^{p_j} w_j/\nu)\|_{L^{1,\infty}(\nu)}^{1/p_j} \\ &\leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)},\end{aligned}$$

and the theorem is proved in the case $p_j > 1$, for every $j = 1, \dots, m$.

In the case where some $p_j = 1$, note that the condition $\left(\fint_Q w_j^{1-p_j'}\right)^{\frac{1}{p_j'}}$ must be understood as $(\operatorname{ess\,inf}_Q w_j)^{-1}$. Indeed, as in the linear case, taking limits we obtain

$$\left(\fint_Q w_j^{-\frac{1}{p_j-1}}(x)dx\right)^{p_j-1} = ||w_j^{-1}||_{L^{\frac{1}{p_j-1}}(Q,|Q|^{-1}dx)} \xrightarrow{p_j\to 1} ||w_j^{-1}||_{L^\infty(Q)}.$$

Let us assume now that $p_j = 1$, $j = 1, \ldots, m$. Using again (4.6) with $p_j = 1$ and $f_j = \chi_S$, for every j and S being a mesurable subset of Q with positive measure, we obtain

$$\nu(Q)^{\frac{1}{p}} \prod_{j=1}^{m} \frac{1}{|Q|} \int_S dx \leq C \prod_{j=1}^{m} \int_S w_j.$$

Since $m = 1/p$, we can write

$$\left(\fint_Q \nu(x) dx \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{m} \frac{1}{|S|} \int_S w_j(x_j) dx_j,$$

for every arbitrary cube Q and $S \subset Q$ measurable set with positive measure. Let $a > \prod_{j=1}^{m} \operatorname{ess\,inf}_Q w_j$, and consider the set

$$S_a = \left\{ x \in Q : \prod_{j=1}^{m} w_j < a \right\}.$$

It is clear that $S_a \subset Q$ and it has positive measure. Therefore,

$$\begin{aligned} \left(\fint_Q \nu(x) dx \right)^{\frac{1}{p}} &\leq C \left(\frac{1}{|S_a|} \int_{S_a} w_1(x_1) dx_1 \right) \cdot \ldots \cdot \left(\frac{1}{|S_a|} \int_{S_a} w_m(x_m) dx_m \right) \\ &= C \int_{S_a} \cdots \int_{S_a} \prod_{j=1}^{m} \frac{w_j(x_j)}{|S_a|} dx_1 \ldots dx_m \leq Ca, \end{aligned}$$

for every $a > \operatorname{ess\,inf}_Q w_j$. Hence,

$$\left(\frac{1}{|Q|} \int_Q \nu(x) dx \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{m} \operatorname{ess\,inf}_Q w_j,$$

for almost every $x \in Q$. Since Q is arbitrary, we obtain

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu \right)^{\frac{1}{p}} \left(\prod_{j=1}^{m} \operatorname{ess\,inf}_Q w_j \right)^{-1} \leq C, \tag{4.9}$$

and we are done. Conversely, assume (4.9). It follows that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu \right)^{\frac{1}{p}} \left(\prod_{j=1}^{m} w_j \right)^{-1} \leq C,$$

and, therefore,

$$\left(\frac{\nu(Q)}{|Q|} \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{m} w_j. \tag{4.10}$$

Assume without loss of generality that $f_j \geq 0$, for every $j = 1, \ldots, m$. Then, using (4.10), we have that

$$\left(\fint_Q \nu \right)^{\frac{1}{p}} \prod_{j=1}^{m} \left(\fint_Q f_j \right) = \prod_{j=1}^{m} \fint_Q f_j \left(\frac{\nu(Q)}{|Q|} \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{m} \fint_Q f_j w_j,$$

for every cube Q, obtaining that we have proved (4.8) in the case $p_j = 1$, for all $j = 1, \dots, m$. From this point on, the argument is similar as in the case when all the indexes $p_j > 1$, so that we can conclude that this condition is also sufficient in this case. In the case when $p_j = 1$, $j = 1, \dots, l$, and $p_j > 1$, $j = l, \dots, m$, with $1 < l < m$ it suffices to combine the previous estimates to get the result and we are done. $\square$

Remark 4.2. By a close inspection of the previous proof if we denote

$$[v, \overrightarrow{w}]_{A_{\overrightarrow{P}}} := \sup_Q \Big(\frac{1}{|Q|}\int_Q \nu\Big) \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{p/p_j'},$$

then the best constant appearing in (4.3) is comparable to $[v, \overrightarrow{w}]_{A_{\overrightarrow{P}}}^{1/p}$. Also observe that condition (4.4) combined with Lebesgue differentiation theorem implies that $\nu(x) \le c \prod_{j=1}^m w_j(x)^{p/p_j}$ *a.e.* This suggests a way to define an analogue of the Muckenhoupt A_p classes in the multiple setting.

5. The $A_{\vec{P}}$ class of weights

Let us now introduce the multiple classes of weights as well as their relationship with the Muckenhoupt's A_p classes of weights and other interesting properties.

Definition 5.1. For m exponents $p_1, \cdots, p_m$, we will often write p for the number given by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\vec{P}$ for the vector $\vec{P} = (p_1, \cdots, p_m)$.

Definition 5.2. Let $1 \le p_1, \dots, p_m < \infty$. Given $\vec{w} = (w_1, \dots, w_m)$, set

$$\nu_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big) \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{p/p_j'} < \infty. \tag{5.1}$$

When $p_j = 1$, $\Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{p/p_j'}$ must be understood as $(\operatorname{ess\,inf}_Q w_j)^{-p}$.

We will refer to (5.1) as the *multilinear $A_{\vec{P}}$ constant.*
It is not difficult to prove by using Hölder's inequality, that $\nu_{\vec{w}} \in A_{mp}$ and

$$\prod_{j=1}^m A_{p_j} \subset A_{\vec{P}}.$$

These results are left as exercises for the reader.
The multiple weight classes can be characterized in terms of the linear A_p classes. Observe that the following theorem also shows that as the index m increases, the $A_{\vec{P}}$ condition gets weaker. It is also possible to show that the two conditions below are independent of each other.

Proposition 5.3. *Let $\vec{w}=(w_1,\cdots,w_m)$ and $1\le p_1,\dots,p_m<\infty$. Then $\vec{w}\in A_{\vec{P}}$ if and only if*

$$\begin{cases} w_j^{1-p_j'}\in A_{mp_j'},\ j=1,\dots,m\\ \nu_{\vec{w}}\in A_{mp}, \end{cases} \tag{5.2}$$

where the condition $w_j^{1-p_j'}\in A_{mp_j'}$ in the case $p_j=1$ is understood as $w_j^{1/m}\in A_1$.

Proof of Proposition 5.3. Consider first the case when there exists at least one $p_j>1$. Without loss of generality we can assume that $p_1,\dots,p_l=1, 0\le l<m$, and $p_j>1$ for $j=l+1,\dots,m$.
Suppose that $\vec{w}$ satisfies the multilinear $A_{\vec{P}}$ condition.
Fix $j\ge l+1$ and define the numbers

$$q_j=p\Big(m-1+\frac{1}{p_j}\Big)\quad\text{and}\quad q_i=\frac{p_i}{p_i-1}\frac{q_j}{p},i\ne j,i\ge l+1.$$

We first prove that $w_j^{1-p_j'}\in A_{mp_j'}$ for $j\ge l+1$, i.e.,

$$\Big(\int_Q w_j^{-1/(p_j-1)}\Big)\Big(\int_Q w_j^{\frac{p}{p_jq_j}}\Big)^{\frac{q_jp_j}{p(p_j-1)}}\le c|Q|^{\frac{mp_j}{p_j-1}}. \tag{5.3}$$

Since

$$\sum_{i=l+1}^m\frac{1}{q_i}=\frac{1}{m-1+1/p_j}\Big(\frac{1}{p}+\sum_{i=l+1,i\ne j}^m(1-1/p_i)\Big)=1,$$

applying the Hölder's inequality, we obtain

$$\begin{aligned}\int_Q w_j^{\frac{p}{p_jq_j}} &= \int_Q\Big(\prod_{i=l+1}^m w_i^{\frac{p}{p_iq_j}}\Big)\Big(\prod_{i=l+1,i\ne j}^m w_i^{-\frac{p}{p_iq_j}}\Big)\\ &\le \Big(\int_Q\prod_{i=l+1}^m w_i^{p/p_i}\Big)^{1/q_j}\prod_{i=l+1,i\ne j}^m\Big(\int_Q w_i^{-1/(p_i-1)}\Big)^{1/q_i}.\end{aligned}$$

From this inequality and the $A_{\vec{P}}$ condition we easily get (5.3).
Next we show that $\nu_{\vec{w}}\in A_{mp}$. Setting $s_j=(m-1/p)p_j'$, $j\ge l+1$, we have $\sum_{j=l+1}^m\frac{1}{s_j}=1$ and, therefore, by Hölder's inequality,

$$\int_Q\prod_{j=l+1}^m w_j^{-\frac{p}{p_j(pm-1)}}\le\prod_{j=l+1}^m\Big(\int_Q w_j^{-1/(p_j-1)}\Big)^{1/s_j}. \tag{5.4}$$

Hence,

$$\int_Q(\nu_{\vec{w}})^{-\frac{1}{pm-1}}\le\prod_{j=1}^l(\operatorname{ess\,inf}_Q w_j)^{-\frac{p}{pm-1}}\prod_{j=l+1}^m\Big(\int_Q w_j^{-1/(p_j-1)}\Big)^{1/s_j}.$$

Combining this inequality with the $A_{\vec{P}}$ condition gives $\nu_{\vec{w}}\in A_{mp}$.

Suppose now that $l>0$, and let us show that $w_j^{1/m}\in A_1, j=1,\dots,l$. Fix $1\le i_0\le l$. By Hölder's inequality and (5.4),

$$\begin{aligned}\int_Q w_{i_0}^{1/m} &\le \Big(\int_Q w_{i_0}^p\prod_{j=l+1}^m w_j^{p/p_j}\Big)^{1/pm}\Big(\int_Q\prod_{j=l+1}^m w_j^{-\frac{p}{p_j(pm-1)}}\Big)^{1-1/pm}\\ &\le \Big(\int_Q w_{i_0}^p\prod_{j=l+1}^m w_j^{p/p_j}\Big)^{1/pm}\prod_{j=l+1}^m\Big(\int_Q w_j^{1-p_j'}\Big)^{\frac{1}{mp_j'}}\end{aligned}$$

This inequality combined with the $A_{\vec{P}}$ condition proves $w_{i_0}^{1/m}\in A_1$. Thus we have proved that $\vec{w}\in A_{\vec{P}}\Rightarrow$ (5.2).

To prove that (5.2) is sufficient for $\vec{w}\in A_{\vec{P}}$, we first observe that for any weight w_j,

$$1\le\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{pm-1}}\Big)^{m-1/p}\prod_{j=1}^m\Big(\frac{1}{|Q|}\int_Q w_j^{\frac{1}{p_j(m-1)+1}}\Big)^{m-1+1/p_j}. \tag{5.5}$$

Indeed, let $\alpha=\frac{1}{1+pm(m-1)}$ and $\alpha_j=\frac{1/p+m(m-1)}{1/p_j+m-1}$. Then $\sum_{j=1}^m 1/\alpha_j=\ 1$, and by Hölder's inequality,

$$\int_Q\nu_{\vec{w}}^\alpha\le\prod_{j=1}^m\Big(\int_Q w_j^{\frac{\alpha p\alpha_j}{p_j}}\Big)^{1/\alpha_j}=\prod_{j=1}^m\Big(\int_Q w_j^{\frac{1}{p_j(m-1)+1}}\Big)^{\alpha p(m-1+1/p_j)}.$$

Using again the Hölder's inequality, we have

$$1\le\Big(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}^\alpha\Big)\Big(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}^{-\frac{1}{pm-1}}\Big)^{\alpha(pm-1)}.$$

This inequality along with the previous one yields (5.5). Finally, (5.5) combined with (5.2) easily gives that $\vec{w}\in A_{\vec{P}}$.

It remains to consider the case when $p_j=1$ for all $j=1,\cdots,m$. Assume that $\vec{w}\in A_{(1,\cdots,1)}$, i.e.,

$$\left(\frac{1}{|Q|}\int_Q\Big(\prod_{j=1}^m w_j\Big)^{1/m}\right)^m\le c\prod_{j=1}^m \operatorname{ess\,inf}_Q w_j. \tag{5.6}$$

It is clear that (5.6) implies that $w_j^{1/m}\in A_1$, $j=1,\dots,m$ and $\nu_{\vec{w}}\in A_1$. Conversely, combining these last conditions with Hölder's inequality we obtain

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q\Big(\prod_{j=1}^m w_j\Big)^{1/m}\right)^m &\le c\operatorname{ess\,inf}_Q\Big(\prod_{j=1}^m w_j\Big)\le c\left(\frac{1}{|Q|}\int_Q\Big(\prod_{j=1}^m w_j\Big)^{1/m^2}\right)^{m^2}.\\ &\le c\prod_{j=1}^m\Big(\frac{1}{|Q|}\int_Q w_j^{1/m}\Big)^m\le c\prod_{j=1}^m\operatorname{ess\,inf}_Q w_j.\end{aligned}$$

This proves that $\vec{w}\in A_{(1,\cdots,1)}$ is equivalent to $w_j^{1/m}\in A_1$, $j=1,\dots,m$ and $\nu_{\vec{w}}\in A_1$.
The theorem is proved. □

Remark 5.4. As a consequence of Proposition 5.3, observe that if $1 < p_j < \infty, j = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ and $\vec{w} \in A_{\vec{P}}$, then

$$[\sigma_j]_{A_\infty} \leq C[\vec{w}]_{A_{\vec{P}}}^{p_j'/p}.$$

Remark 5.5. Observe that $A_{(1,\cdots,1)}$ is contained in $A_{\vec{P}}$ for each $\vec{P}$, however the classes $A_{\vec{P}}$ are not increasing with the natural partial order. Indeed, consider the partial order relation between vectors $\vec{P} = (p_1, \ldots, p_m)$ and $\vec{Q} = (q_1, \ldots, q_m)$ given by

$$\vec{P} \lesssim \vec{Q} \quad \text{si} \quad p_j \leq q_j \quad \forall j = 1, \ldots, m.$$

Then, since the A_p classes are increasing, we can write

$$\prod_{j=1}^m A_{p_j} \subseteq \prod_{j=1}^m A_{q_j}.$$

However $A_{\vec{P}}$ is not contained in $A_{\vec{Q}}$. To see this, consider $n = 1$, $m = 2$, $\vec{P} = (2, 2)$ and the vector of weights $\vec{w} = (w_1, w_2) = (|x|^{-\frac{5}{3}}, 1)$. We now need to check that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w_1^{1/2} \right) \left(\frac{1}{|Q|} \int_Q w_1^{-1} \right)^{1/2} < \infty.$$

Since $w_1^{1/2} \in A_1$ since it is a power weight of the form $|x|^\alpha$ such that $-n < \alpha < n(p-1)$, we can write

$$\begin{aligned}
\left(\frac{1}{|Q|} \int_Q w_1^{1/2} \right) \left(\frac{1}{|Q|} \int_Q w_1^{-1} \right)^{1/2} &\lesssim (\operatorname{ess\,inf}_Q w_1)^{1/2} \left(\frac{1}{|Q|} \int_Q w_1^{-1} \right)^{1/2} \\
&= \left(\frac{1}{|Q|} \int_Q w_1^{-1} \operatorname{ess\,inf}_Q w_1 \right)^{1/2} \\
&\leq \left(\frac{1}{|Q|} \int_Q w_1^{-1} w_1 \right)^{1/2} < \infty.
\end{aligned}$$

Therefore, $\vec{w} \in A_{(2,2)}$. However, w_1 raised to an appropriate large power becomes non-locally integrable and, it is easy to show that $\vec{w} \notin A_{\vec{Q}}$ when, for instance, $\vec{Q} = (2, 6)$. In fact, if $\vec{w} \in A_{(2,6)}$ we would need to verify that the following condition holds

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w_1^{3/4} \right)^{2/3} \left(\frac{1}{|Q|} \int_Q w_1^{-1} \right)^{1/2}.$$

Since $w_1^{3/4}$ is not locally integrable, the quantity above is not finite and $\vec{w} \notin A_{(2,6)}$.

Remark 5.6. The condition $\vec{w} \in A_{\vec{P}}$ does not imply in general $w_j \in L^1_{\text{loc}}$ for any j. Take, for instance,

$$w_1 = \frac{\chi_{[0,2]}(x)}{|x-1|} + \chi_{\mathbb{R}/[0,2]}(x)$$

and $w_j(x) = \frac{1}{|x|}$ for $j = 2, ..., m$. Then, using the definition, it is not difficult to check that $\nu_{\vec{w}} \in A_1$. We also have $\operatorname{ess\,inf}_Q \nu_{\vec{w}} \sim \prod_{j=1}^m \operatorname{ess\,inf}_Q w_j^{p/p_j}$. These last two facts together easily imply that $\vec{w} \in A_{\vec{P}}$.

6. Sharp mixed bounds for the multilinear maximal function

In [37, Thm. 3.7] was proved that $A_{\vec{P}}$ is a necessary and sufficient condition for the boundedness of the multilinear maximal function from an appropriate product of weighted Lebesgue spaces into $L^p(\nu_{\vec{w}})$. Here we are also going to prove the sharp bounds for $\mathcal{M}$ which extend the linear results contained in [25] and [5].
It is clear that the $A_{\vec{P}}$ condition is necessary for the strong boundedness of $\mathcal{M}$ as a consequence of Theorem 4.1. Now, we are going to prove that this condition is also sufficient and, by the way, we obtain a sharp mixed bound whose original proof can be found in [15].

Theorem 6.1. *Let $1 < p_i < \infty, i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\vec{P}}\, [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \prod_{i=1}^{m}([\sigma_i]_{A_\infty})^{\frac{1}{p_i}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}, \tag{6.1}$$

holds if $\vec{w} \in A_{\vec{P}}$, where $\sigma_i = w_i^{1-p_i'}$, $i = 1, \ldots, m$. Furthermore the exponents are sharp in the sense that they cannot be replaced by smaller ones.

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

Denote the standard grid by $\mathcal{D}$. By a *general dyadic grid* $\mathscr{D}$ we mean a collection of cubes with the following properties:

- **(i):** For any $Q \in \mathscr{D}$ its sidelength ℓ_Q is of the form $2^k, k \in \mathbb{Z}$.
- **(ii):** $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathscr{D}$.
- **(iii):** The cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

We say that $\{Q_j^k\}$ is a *sparse family* of cubes if:

- **(i):** the cubes Q_j^k are disjoint in j, with k fixed;
- **(ii):** if $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$;
- **(iii):** $|\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2}|Q_j^k|$.

With each sparse family $\{Q_j^k\}$ we associate the sets $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Observe that the sets E_j^k are pointwise disjoint and $|Q_j^k| \leq 2|E_j^k|$.
First, we will need two lemmas. The first one can be found in [25].

Proposition 6.2. *There are 2^n dyadic grids $\mathscr{D}_\alpha$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_\alpha \in \mathscr{D}_\alpha$ such that $Q \subset Q_\alpha$ and $\ell_{Q_\alpha} \leq 6\ell_Q$.*

Lemma 6.3. *For any non-negative integrable $f_i, i = 1, \ldots, m$, there exist sparse families $\mathcal{S}_\alpha \in \mathscr{D}_\alpha$ such that for all $x \in \mathbb{R}^n$,*

$$\mathcal{M}(\overrightarrow{f})(x) \leq (2 \cdot 12^n)^m \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathscr{D}_\alpha, \mathcal{S}_\alpha}(\overrightarrow{f})(x),$$

where $\overrightarrow{f} = (f_1, \ldots, f_m)$ and given a sparse family $\mathcal{S} = \{Q_j^k\}$ of cubes from a dyadic grid $\mathscr{D}$, the operator $\mathcal{A}_{\mathscr{D},\mathcal{S}}$ is given by

$$\mathcal{A}_{\mathscr{D},\mathcal{S}}(\overrightarrow{f}) = \sum_{j,k} \left(\prod_{i=1}^{m} (f_i)_{Q_j^k} \right) \chi_{Q_j^k}.$$

Proof of Lemma 6.3. First, by Proposition 6.2,

$$\mathcal{M}(\vec{f})(x) \le 6^{mn} \sum_{\alpha=1}^{2^n} \mathcal{M}^{\mathscr{D}_\alpha}(\vec{f})(x), \tag{6.2}$$

where $\mathcal{M}^{\mathscr{D}_\alpha}$ denotes the multilinear maximal function defined with respect to $\mathscr{D}_\alpha$. Consider $\mathcal{M}^d(\vec{f})$ taken with respect to the standard dyadic grid. We will use exactly the same argument as in the Calderón-Zygmund decomposition. For c_n which will be specified below and for $k \in \mathbb{Z}$ consider the sets

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > c_n^k\}.$$

Then we have that $\Omega_k = \cup_j Q_j^k$, where the cubes Q_j^k are pairwise disjoint with k fixed, and

$$c_n^k < \prod_{i=1}^{m} (f_i)_{Q_j^k} \le 2^{mn} c_n^k.$$

From this and from Hölder's inequality,

$$\begin{aligned} |Q_j^k \cap \Omega_{k+1}| &= \sum_{Q_l^{k+1} \subset Q_j^k} |Q_l^{k+1}| \\ &< c_n^{-\frac{k+1}{m}} \sum_{Q_l^{k+1} \subset Q_j^k} \prod_{i=1}^{m} \Big(\int_{Q_l^{k+1}} f_i \Big)^{1/m} \\ &\le c_n^{-\frac{k+1}{m}} \prod_{i=1}^{m} \Big(\int_{Q_j^k} f_i \Big)^{1/m} \le 2^n c_n^{-1/m} |Q_j^k|. \end{aligned}$$

Hence, taking $c_n = 2^{m(n+1)}$, we obtain that the family $\{Q_j^k\}$ is sparse, and

$$\mathcal{M}^d(\vec{f})(x) \le 2^{m(n+1)} \mathcal{A}_{\mathcal{D},\mathcal{S}}(\vec{f})(x).$$

Applying the same argument to each $\mathcal{M}^{\mathscr{D}_\alpha}(\vec{f})$ and using (6.3), we get the statement of the lemma. □

Next we proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. By Proposition 6.2, it follows

$$\mathcal{M}(\vec{f})(x) \le 6^{mn} \sum_{\alpha=1}^{2^n} \mathcal{M}^{\mathscr{D}_\alpha}(\vec{f})(x), \tag{6.3}$$

where $\mathcal{M}^{\mathscr{D}_\alpha}$ denotes the multilinear maximal function defined with respect to $\mathscr{D}_\alpha$. Then, it suffices to prove the theorem for the dyadic maximal operators $\mathcal{M}^{\mathscr{D}_\alpha}$. Since the proof is independent of the particular dyadic grid, without loss of generality we consider $\mathcal{M}^d$ taken with respect to the standard dyadic grid $\mathcal{D}$.

Let $a = 2^{m(n+1)}$. and $\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > a^k\}$. We have seen in the proof of Lemma 6.3 that $\Omega_k = \cup_j Q_j^k$, where the family $\{Q_j^k\}$ is sparse and $a^k < \prod_{i=1}^m \frac{1}{|Q_j^k|}\int_{Q_j^k}|f_i| \le 2^{nm}a^k$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p\, \nu_{\vec{w}} dx &= \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} \mathcal{M}^d(\vec{f})^p\, \nu_{\vec{w}} dx \\
&\le a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|}\int_{Q_j^k} |f_i| dy_i\right)^p \nu_{\vec{w}}(Q_j^k) \\
&\le a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|}\int_{Q_j^k} |f_i| w_i^{\frac{1}{p_i}} w_i^{-\frac{1}{p_i}} dy_i\right)^p \nu_{\vec{w}}(Q_j^k) \\
&\le a^p \sum_{k,j}\prod_{i=1}^m \left(\frac{1}{|Q_j^k|}\int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i\right)^{\frac{p}{\alpha_i}} \left(\frac{1}{|Q_j^k|}\int_{Q_j^k} w_i^{-\frac{\alpha_i'}{p_i}} dy_i\right)^{\frac{p}{\alpha_i'}} \nu_{\vec{w}}(Q_j^k),
\end{aligned}$$

where $\alpha_i = (p_i' r_i)'$ and r_i is the exponent in the sharp reverse Hölder inequality (see [25, Thm. 2.3 (a)]) for the weights σ_i which are in A_∞ for $i = 1, \dots, m$. Applying the sharp Reverse Hölder inequality for each σ_i, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p\, \nu_{\vec{w}} dx &\le a^p \sum_{k,j}\prod_{i=1}^m \left(\frac{1}{|Q_j^k|}\int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i\right)^{\frac{p}{\alpha_i}} \\
&\quad \times \left(2\frac{1}{|Q_j^k|}\int_{Q_j^k} \sigma_i\right)^{\frac{p}{p_i'}} \nu_{\vec{w}}(Q_j^k) \\
&\le C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j}\prod_{i=1}^m \left(\frac{1}{|Q_j^k|}\int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i\right)^{\frac{p}{\alpha_i}} |Q_j^k|.
\end{aligned}$$

Let E_j^k be the sets associated with the family $\{Q_j^k\}$. Using the properties of E_j^k and Hölder's inequality with the exponents p_i/p, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p\, \nu_{\vec{w}} dx &\le 2C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j}\prod_{i=1}^m \left(\frac{1}{|Q_j^k|}\int_{Q_j^k} |f_i(y_i)|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i\right)^{\frac{p}{\alpha_i}} |E_j^k| \\
&\le 2C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \int_{E_j^k} \prod_{i=1}^m M\left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}}\right)^{\frac{p}{\alpha_i}} dx \\
&\le 2C[\vec{w}]_{A_{\vec{P}}} \int_{\mathbb{R}^n} \prod_{i=1}^m M\left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}}\right)^{\frac{p}{\alpha_i}} dx \\
&\le 2C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} M\left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}}\right)^{\frac{p_i}{\alpha_i}} dx\right)^{\frac{p}{p_i}}.
\end{aligned}$$

From this and by the boundedness of M,

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \,\nu_{\vec{w}} dx &\le C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m \big((p_i/\alpha_i)'\big)^{\frac{p}{p_i}} \Big\| |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \Big\|_{L^{\frac{p_i}{\alpha_i}}(\mathbb{R}^n)}^{\frac{p}{\alpha_i}} \\ &\le C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m (p_i' r_i')^{\frac{p}{p_i}} \|f_i\|_{L^{p_i}(w_i)}^p \\ &\le C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m ([\sigma_i]_{A_\infty})^{\frac{p}{p_i}} \|f_i\|_{L^{p_i}(w_i)}^p,\end{aligned}$$

where in next to last inequality we have used that $(p_i/\alpha_i)' \le p_i' r_i'$ and in the last inequality we have used that $r_i' \approx [\sigma_i]_{A_\infty}$, for $i = 1, \dots, m$. This completes the proof of (6.1). □

Let us show now the sharpness of the exponents in (6.1). Assume that $n = 1$ and $0 < \varepsilon < 1$. Let

$$w_i(x) = |x|^{(1-\varepsilon)(p_i-1)} \quad \text{and} \quad f_i(x) = x^{-1+\varepsilon}\chi_{(0,1)}(x), \quad i = 1, \dots, m.$$

On one hand, it is easy to check that $\nu_{\vec{w}} = |x|^{(1-\varepsilon)(pm-1)}$ and

$$[\vec{w}]_{A_{\vec{P}}} = [\nu_{\vec{w}}]_{A_{pm}} \approx (1/\varepsilon)^{mp-1}. \tag{6.4}$$

We also need to estimate $[\sigma_i]_{A_\infty}$, for $i = 1, \dots, m$. We have that

$$\sigma_i = w_i^{1-p_i'} = |x|^{\varepsilon-1} := \sigma.$$

Since σ is a power weight belonging to the A_1 class of weights, we obtain

$$[\sigma]_{A_\infty} \le [\sigma]_{A_1} \approx \frac{1}{\varepsilon}. \tag{6.5}$$

Hence

$$\prod_{i=1}^m [\sigma]_{A_\infty}^{\frac{1}{p_i}} = [\sigma]_{A_\infty}^{\frac{1}{p}} = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}. \tag{6.6}$$

Besides,

$$\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)} = (1/\varepsilon)^{1/p}. \tag{6.7}$$

On the other hand, we need to estimate $||\mathcal{M}(\vec{f})||_{L^p(\nu_{\vec{w}})}$. First, let $f = x^{-1+\varepsilon}\chi_{(0,1)}(x)$ and observe that

$$||\mathcal{M}(\vec{f})||_{L^p(\nu_{\vec{w}})} = ||Mf||_{L^{pm}(\nu_{\vec{w}})}^m$$

and if we pick $0 < x < 1$, we obtain

$$Mf(x) \ge \frac{1}{x}\int_0^x y^{-1+\varepsilon} dy = \frac{f(x)}{\varepsilon}.$$

Then the left-hand side of (6.1) can be bounded from below as follows:

$$
\begin{aligned}
||\mathcal{M}(\vec{f})||_{L^p(\nu_{\vec{w}})} = ||Mf||^m_{L^{pm}(\nu_{\vec{w}})} &\geq \left(\frac{1}{\varepsilon}\right)^m \left(\int_{\mathbb{R}} f(x)^{mp}\nu_{\vec{w}}\right)^{\frac{m}{mp}} \\
&= \left(\frac{1}{\varepsilon}\right)^m ||f||^m_{L^{pm}(\nu_{\vec{w}})} \\
&\approx \left(\frac{1}{\varepsilon}\right)^m \left(\frac{1}{\varepsilon}\right)^{1/p} \\
&\geq \left(\frac{1}{\varepsilon}\right)^{m+1/p}.
\end{aligned} \tag{6.8}
$$

since

$$||f||^{mp}_{L^{pm}(\nu_{\vec{w}})} \approx \frac{1}{\varepsilon},$$

and $\nu_{\vec{w}} \in A_{pm}$. By (6.4), (6.6) and (6.7) the right-hand side of (6.1) is at most $(1/\varepsilon)^{m+1/p}$. Since ε is arbitrary, this shows that the exponents $1/p$ and $1/p_i$ on the right-hand side of (6.1) cannot be replaced by smaller ones.

7. Multilinear Sawyer's theorem

In this section, we introduce a multilinear nonstandard formulation of the (dyadic) Carleson embedding theorem originally proved in [25]. This result was the key lemma to prove in [9] a generalization of Sawyer's two weight theorem for the multisublinear maximal function $\mathcal{M}$. Some remarks as well as some recent advances in this problem are listed within the section.

Lemma 7.1. *Suppose that the nonnegative numbers $\{a_Q\}_Q$ satisfy*

$$\sum_{Q\subset R} a_Q \leq A\int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\,dx,\ \forall R\in\mathscr{D} \tag{7.1}$$

where σ_i are weights for $i = 1,\dots,m$. Then for all $1 < p_i < \infty$ and $p \in (1,\infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and for all $f_i \in L^{p_i}(\sigma_i)$,

$$
\begin{aligned}
\left(\sum_{Q\in\mathscr{D}} a_Q\Big(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q f_i(y_i)\sigma_i(y_i)dy_i\Big)^p\right)^{1/p} &\leq A||\mathcal{M}^d_{\vec{\sigma}}(\overrightarrow{f})||_{L^p(\nu_{\vec{\sigma}})} \\
&\leq A\prod_{i=1}^m p_i'||f_i||_{L^{p_i}(\sigma_i)},
\end{aligned} \tag{7.2}
$$

where $\mathcal{M}^d_{\vec{\sigma}}(\overrightarrow{f}) = \sup\limits_{\substack{Q\ni x\\ Q\in\mathscr{D}}} \prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i(y_i)|\sigma_i(y_i)dy_i.$

Proof of Lemma 7.1. Let us see the sum

$$\sum_{Q\in\mathscr{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q f_i(y_i)\sigma_i(y_i)dy_i\right)^p$$

as an integral on a measure space $(\mathscr{D}, 2^{\mathscr{D}}, \mu)$ built over the set of dyadic cubes $\mathscr{D}$, assigning to each $Q \in \mathscr{D}$ the measure a_Q. Thus

$$\begin{aligned}
&\sum_{Q\in\mathscr{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q f_i(y_i)\sigma_i(y_i)dy_i\right)^p = \\
&= \int_0^\infty p\lambda^{p-1}\mu\left\{Q\in\mathscr{D} : \prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q f_i(y_i)\sigma_i(y_i)dy_i > \lambda\right\}\\
&=: \int_0^\infty p\lambda^{p-1}\mu(\mathscr{D}_\lambda)d\lambda.
\end{aligned}$$

Let us denote by $\mathscr{D}_\lambda^*$ the set of maximal dyadic cubes R with the property that $\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_R f_i(y_i)\sigma_i(y_i)dy_i > \lambda$. Then the cubes $R \in \mathscr{D}_\lambda^*$ are disjoint and their union is equal to the set $\{\mathcal{M}^d_{\vec{\sigma}}(\vec{f}) > \lambda\}$. Thus

$$\begin{aligned}
\mu(\mathscr{D}_\lambda) &= \sum_{Q\in\mathscr{D}_\lambda} a_Q \le \sum_{R\in\mathscr{D}_\lambda^*}\sum_{Q\subset R} a_Q\\
&\le A\sum_{R\in\mathscr{D}_\lambda^*}\int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}dx\\
&= A\int_{\{\mathcal{M}^d_{\vec{\sigma}}(\vec{f})>\lambda\}} \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}dx.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\sum_{Q\in\mathscr{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q f_i(y_i)\sigma_i(y_i)dy_i\right)^p &\le A\int_0^\infty p\lambda^{p-1}\int_{\{\mathcal{M}^d_{\vec{\sigma}}(\vec{f})>\lambda\}} \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}dxd\lambda\\
&= A\int_{\mathbb{R}^n} \mathcal{M}^d_{\vec{\sigma}}(\vec{f})^p \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}dx\\
&\le A\int_{\mathbb{R}^n} \prod_{i=1}^m ((M^d_{\sigma_i}(f_i))^{p_i}\sigma_i)^{\frac{p}{p_i}}dx\\
&\le A\prod_{i=1}^m \left(\int_{\mathbb{R}^n} (M^d_{\sigma_i}(f_i))^{p_i}\sigma_i dx\right)^{\frac{p}{p_i}}\\
&\le A\prod_{i=1}^m (p_i')^p \left(\int_{\mathbb{R}^n} |f_i|^{p_i}\sigma_i dx\right)^{\frac{p}{p_i}},
\end{aligned}$$

where we have used that $\mathcal{M}^d_{\vec{\sigma}}(\overrightarrow{f}) \leq \prod_{i=1}^m M^d_{\sigma_i}(f_i)$, Hölder's inequality and the boundedness properties of $M^d_{\sigma_i}(f_i)$ in $L^{p_i}(\sigma_i)$. □

Next we establish the following generalization of Sawyer's theorem for which it is necessary to define the Sawyer's condition in the multilinear setting.

Definition 7.2. We say that the pair $(v, \vec{w})$ satisfies the $S_{\vec{P}}$ condition if

$$[v, \vec{w}]_{S_{\vec{P}}} = \sup_Q \Big(\int_Q \mathcal{M}(\sigma \vec{\chi}_Q)^p v dx \Big)^{\frac{1}{p}} \Big(\prod_{i=1}^m \sigma_i(Q)^{\frac{1}{p_i}} \Big)^{-1} < \infty,$$

where $\sigma \vec{\chi}_Q = (\sigma_1 \chi_Q, \ldots, \sigma_m \chi_Q)$ and $\sigma_i = w_i^{1-p_i'}$ for all $i = 1, \ldots, m$ and all the suprema in the above definitions are taken over all cubes Q in $\mathbb{R}^n$.

Very recently it was shown in [41] a multilinear version of Sawyer's theorem using a kind of monotone property on the weights. The condition that we establish here is a sort of reverse Hölder inequality in the multilinear setting.

Definition 7.3. We say that the vector $\vec{w}$ satisfies the $RH_{\vec{P}}$ condition if there exists a positive constant C such that

$$\prod_{i=1}^m \Big(\int_Q \sigma_i dx \Big)^{\frac{p}{p_i}} \leq C \int_Q \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \tag{7.3}$$

where $\sigma_i = w_i^{1-p_i'}$ for $i = 1, \ldots, m$. We denote by $[\vec{w}]_{RH_{\vec{P}}}$ the smallest constant C in (7.3).

Observe that when $m = 1$ this reverse Hölder condition is superfluous and we recover the linear result of Moen in [43].

Theorem 7.4. *Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. If we suppose that $\overrightarrow{w} \in RH_{\overrightarrow{P}}$ then there exists a positive constant C such that*

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \leq C \prod_{i=1}^m ||f_i||_{L^{p_i}(\sigma_i)}, \;\; f_i \in L^{p_i}(\sigma_i), \tag{7.4}$$

where $\sigma_i = w_i^{1-p_i'}$, if and only if $(v, \overrightarrow{w}) \in S_{\overrightarrow{P}}$. Moreover, if we denote the smallest constant C in (7.4) by $||\mathcal{M}||$, we obtain

$$[v, \overrightarrow{w}]_{S_{\overrightarrow{P}}} \lesssim ||\mathcal{M}|| \lesssim [v, \overrightarrow{w}]_{S_{\overrightarrow{P}}} [\overrightarrow{w}]^{1/p}_{RH_{\overrightarrow{P}}}. \tag{7.5}$$

Here we make some remarks related to the previous theorem.

Remark 7.5. In the particular case when $v = \nu_{\overrightarrow{w}}$, the following statements are equivalent:

(1) $\overrightarrow{w} \in A_{\overrightarrow{P}}$.
(2) $\sigma_i = w_i^{1-p_i'} \in A_{mp_i'}$, for $i = 1, \ldots, m$ and $\nu_{\overrightarrow{w}} \in A_{mp}$.
(3) $(\nu_{\overrightarrow{w}}, \overrightarrow{w}) \in S_{\overrightarrow{P}}$.

(4) There exists a positive constant C such that

$$||\mathcal{M}(\overrightarrow{f})||_{L^p(\nu_{\overrightarrow{w}})} \leq C\prod_{i=1}^m ||f_i||_{L^{p_i}(w_i)},\ f_i\in L^{p_i}(w_i). \tag{7.6}$$

Indeed, the equivalence between 1., 2. and 4. was proved in [37, Th. 3.6, Th. 3.7]. It can be easily seen that in this particular case $[\nu_{\overrightarrow{w}},\overrightarrow{w}]_{S_{\overrightarrow{P}}}\lesssim ||\mathcal{M}||$ where $||\mathcal{M}||$ denotes the smallest constant in (7.6) and $[\overrightarrow{w}]_{A_{\overrightarrow{P}}}\lesssim [\nu_{\overrightarrow{w}},\overrightarrow{w}]^p_{S_{\overrightarrow{P}}}$. Therefore we have that 4. implies 3. and 3. implies 1.. So we have obtained that all the statements are equivalent.

Additionally, following Theorem 6.1 we also have that $||\mathcal{M}||\lesssim [\overrightarrow{w}]^{1/p}_{A_{\overrightarrow{P}}}\prod_{i=1}^m[\sigma_i]_\infty^{\frac{1}{p_i}}$. So, we have obtained

$$[\overrightarrow{w}]^{1/p}_{A_{\overrightarrow{P}}}\lesssim [v_{\overrightarrow{w}},\overrightarrow{w}]_{S_{\overrightarrow{P}}}\lesssim ||\mathcal{M}||\lesssim [\overrightarrow{w}]^{1/p}_{A_{\overrightarrow{P}}}\prod_{i=1}^m[\sigma_i]_\infty^{\frac{1}{p_i}}. \tag{7.7}$$

Remark 7.6. As we have observed in the previous remark, $RH_{\overrightarrow{P}}$ condition is not necessary when $v=\nu_{\overrightarrow{w}}$ in Theorem 7.4. We are not sure if this condition can be removed in the general case.

Proof of Theorem 7.4. It is clear that (7.4) implies the $S_{\overrightarrow{P}}$ condition without using that $(v,\overrightarrow{w})\in RH_{\overrightarrow{P}}$. Thus, it remains to prove that $(v,\overrightarrow{w})\in S_{\overrightarrow{P}}$ implies (7.4) to complete the proof of the theorem.
As we did before it suffices to prove the theorem for the dyadic maximal operators $\mathcal{M}^{\mathscr{D}_\alpha}$. Since the proof is independent of the particular dyadic grid, without loss of generality we consider $\mathcal{M}^d$ taken with respect to the standard dyadic grid $\mathcal{D}$. Next we proceed as in the proof of Lemma 6.3. Let $a=2^{m(n+1)}$ and for $k\in\mathbb{Z}$ consider the following sets

$$\Omega_k=\{x\in\mathbb{R}^n:\mathcal{M}^d(\overrightarrow{f\sigma})>a^k\}.$$

Then we have that $\Omega_k=\cup_jQ_j^k$, where the cubes Q_j^k are pairwise disjoint with k fixed, and

$$a^k<\prod_{i=1}^m\frac{1}{|Q_j^k|}\int_{Q_j^k}|f_i(y_i)|\sigma_i(y_i)dy_i\leq 2^{mn}a^k.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^n}\mathcal{M}^d(\overrightarrow{f\sigma})^pvdx&=\sum_k\int_{\Omega_k\setminus\Omega_{k+1}}\mathcal{M}^d(\overrightarrow{f\sigma})^pvdx\\&\leq a^p\sum_k\int_{\Omega_k\setminus\Omega_{k+1}}a^{kp}vdx\\&=a^p\sum_{k,j}a^{kp}v(E_j^k),\end{aligned}$$

since $\Omega_k\setminus\Omega_{k+1}=\cup_jE_j^k$ where the sets E_j^k are the sets associated with the family $\{Q_j^k\}$. Then, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx &\le a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p v(E_j^k) \\ &= a^p \sum_{k,j} v(E_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \left(\prod_{i=1}^m \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p \\ &= a^p \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p,\end{aligned}$$

where $a_Q = v(E(Q)) \left(\prod_{i=1}^m \frac{\sigma_i(Q)}{|Q|} \right)^p$, if $Q = Q_j^k$ for some (k,j) where $E(Q)$ denotes the corresponding set E_j^k associated to Q_j^k, and $a_Q = 0$ otherwise. If we apply the Carleson embedding to these a_Q, we will find the desired result provided that

$$\sum_{Q \subset R} a_Q \le A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \; R \in \mathcal{D}.$$

For $R \in \mathcal{D}$, we obtain

$$\begin{aligned}\sum_{Q \subset R} a_Q &= \sum_{Q_j^k \subset R} v(E_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \\ &= \sum_{Q_j^k \subset R} \int_{E_j^k} \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p v(x) dx \\ &\le \sum_{Q_j^k \subset R} \int_{E_j^k} (\mathcal{M}(\overrightarrow{\sigma \chi_R}))^p v(x) dx \\ &\le [v, \vec{w}]_{S_{\vec{P}}}^p \prod_{i=1}^m \sigma_i(R)^{\frac{p}{p_i}} \\ &\le [v, \vec{w}]_{S_{\vec{P}}}^p [\vec{\omega}]_{RH_{\vec{P}}} \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx,\end{aligned}$$

where in the next to last inequality we have used the $S_{\vec{P}}$ condition and in the last inequality we have used the $RH_{\vec{P}}$ condition. Thus, by Lemma 7.1 we get the desired result and the proof is complete.

□

Remark 7.7. In [39], the authors studied the characterization of the two-weight inequality for the fractional version of the multilinear maximal function

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\alpha/mn}} \int_Q |f_i(y_i)| dy_i, \quad 0 \le \alpha < mn,$$

in terms of the multilinear $S_{\vec{P},q}$ condition. However, the result proved in [39, Thm. 1.1] does not hold in the case $\alpha = 0$, which corresponds to the case of the multilinear maximal function.
In the general case, they gave two testing conditions (see [39, Thm. 4.1] for further details) which are equivalent to $\mathcal{M}_\alpha$ to be bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^q(v)$, for weights v, w_1, w_2, $0 \le \alpha < 2n$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/p - \alpha/n$.

8. Sharp bounds for multilinear sparse operators

In this section, we prove some useful results that we are going to use in the second part of this course. More precisely, we determine the sharp bound for multilinear sparse operators as it was shown in [38]. These operators, as we will see in the following, control pointwisely multilinear Calderón–Zygmund operators. We refer the interested reader to Section 9 for a detailed description on the chronological advances this problem.
Firstly, we prove the following symmetry property of $A_{\vec{P}}$ weights.

Lemma 8.1. *Suppose that $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P}}$ and that $1 < p, p_1, \cdots, p_m < \infty$ with $1/p_1 + \cdots + 1/p_m = 1/p$. Then $\vec{w}^i := (w_1, \cdots, w_{i-1}, v_{\vec{w}}^{1-p'}, w_{i+1}, \cdots, w_m) \in A_{\vec{P}^i}$ with $\vec{P}^i = (p_1, \cdots, p_{i-1}, p', p_{i+1}, \cdots, p_m)$ and*

$$[\vec{w}^i]_{A_{\vec{P}^i}} = [\vec{w}]_{A_{\vec{P}}}^{p_i'/p}, \tag{8.1}$$

where

$$[\vec{w}^i]_{A_{\vec{P}^i}} := \sup_Q \left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right) \cdot \left(\frac{1}{|Q|}\int_Q (v_{\vec{w}}^{1-p'})^{1-p}\right)^{p_i'/p} \prod_{\substack{j=1\\ j\neq i}}^m \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{p_i'/p_j'}. \tag{8.2}$$

Proof. Without loss of generality we will only prove the conclusion for $i = 1$. Notice that

$$1/p' + 1/p_2 + \cdots + 1/p_m = 1/p_1'$$

and

$$v_{\vec{w}}^{(1-p')p_1'/p'} \cdot w_2^{p_1'/p_2} \cdots w_m^{p_1'/p_m} = w_1^{1-p_1'}.$$

By the definition of multiple $A_{\vec{P}}$ constant, we have

$$\begin{aligned}[\vec{w}^1]_{A_{\vec{P}^1}} &= \sup_Q \left(\frac{1}{|Q|}\int_Q w_1^{1-p_1'}\right) \cdot \left(\frac{1}{|Q|}\int_Q (v_{\vec{w}}^{1-p'})^{1-p}\right)^{p_1'/p} \\ &\qquad \times \prod_{i=2}^m \left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right)^{p_1'/p_i'} \\ &= [\vec{w}]_{A_{\vec{P}}}^{p_1'/p}.\end{aligned}$$

□

Let us state and prove the main theorem in this section.

Theorem 8.2. *Suppose that $1 < p_1, \cdots, p_m < \infty$ with $1/p_1 + \cdots + 1/p_m = 1/p$ and $\vec{w} \in A_{\vec{P}}$. Then*

$$\|A_{\mathscr{D},\mathcal{S}}(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max(1,\frac{p_1'}{p},\ldots,\frac{p_m'}{p})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Proof. Without loss of generality we may assume that $f_i \geq 0$. We first consider the case when $\frac{1}{m} < p \leq 1$. It suffices to prove that

$$\|\mathcal{A}_{\mathscr{D},\mathcal{S}}(\vec{f}\sigma)\|_{L^p(v_{\vec{w}})} \leq C_{m,n,\vec{P}}[\vec{w}]_{A_{\vec{P}}}^{\max_i(\frac{p_i'}{p})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)},$$

where $\sigma_i = w_i^{1-p_i'}$, $A_{\mathscr{D},\mathcal{S}}(\vec{f}\sigma) = A_{\mathscr{D},\mathcal{S}}(f_1\sigma_1, \cdots, f_m\sigma_m)$. Without loss of generality, we can assume that $p_1 = \min\{p_1, \cdots, p_m\}$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{A}_{\mathscr{D},\mathcal{S}}(\vec{f}\sigma)^p v_{\vec{w}} &\lesssim \sum_{Q\in\mathcal{S}} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q |f_i|\sigma_i\Big)^p v_{\vec{w}}(Q) \\
&= \sum_{Q\in\mathcal{S}} \frac{v_{\vec{w}}(Q)^{p_1'} \prod_{i=1}^m \sigma_i(Q)^{pp_1'/p_i'}}{|Q|^{mpp_1'}} \Big(\prod_{i=1}^m \int_Q |f_i|\sigma_i\Big)^p \\
&\quad \cdot \frac{|Q|^{mp(p_1'-1)}}{v_{\vec{w}}(Q)^{p_1'-1}\prod_{i=1}^m \sigma_i(Q)^{pp_1'/p_i'}} \\
&\leq [\vec{w}]_{A_{\vec{P}}}^{p_1'} \sum_{Q\in\mathcal{S}} \frac{2^{mp(p_1'-1)}|E_Q|^{mp(p_1'-1)}}{v_{\vec{w}}(Q)^{p_1'-1}\prod_{i=1}^m \sigma_i(Q)^{pp_1'/p_i'}} \cdot \Big(\prod_{i=1}^m \int_Q |f_i|\sigma_i\Big)^p.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
|E_Q| &= \int_{E_Q} v_{\vec{w}}^{\frac{1}{mp}} \sigma_1^{\frac{1}{mp_1'}} \cdots \sigma_m^{\frac{1}{mp_m'}} \\
&\leq v_{\vec{w}}(E_Q)^{\frac{1}{mp}} \sigma_1(E_Q)^{\frac{1}{mp_1'}} \cdots \sigma_m(E_Q)^{\frac{1}{mp_m'}}.
\end{aligned} \tag{8.3}$$

Therefore,

$$|E_Q|^{mp(p_1'-1)} \leq v_{\vec{w}}(E_Q)^{p_1'-1}\sigma_1(E_Q)^{\frac{p(p_1'-1)}{p_1'}} \cdots \sigma_m(E_Q)^{\frac{p(p_1'-1)}{p_m'}}$$

and

$$\frac{p(p_1'-1)}{p_i'} - \frac{p}{p_i} = \frac{pp_1'}{p_i'} - p \geq 0.$$

Since $E_Q \subset Q$, we have

$$v_{\vec{w}}(E_Q)^{p_1'-1} \leq v_{\vec{w}}(Q)^{p_1'-1}$$

and hence

$$\sigma_i(E_Q)^{\frac{p(p_1'-1)}{p_i'} - \frac{p}{p_i}} \leq \sigma_i(Q)^{\frac{pp_1'}{p_i'} - p}, \quad i = 1, \cdots, m.$$

It follows that

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}} \frac{|E_Q|^{mp(p_1'-1)}}{v_{\vec{w}}(Q)^{p_1'-1}\prod_{i=1}^m \sigma_i(Q)^{pp_1'/p_i'}} \cdot \Big(\prod_{i=1}^m \int_Q |f_i|\sigma_i\Big)^p \\
&\leq \sum_{Q\in\mathcal{S}} \prod_{i=1}^m \Big(\frac{1}{\sigma_i(Q)}\int_Q |f_i|\sigma_i\Big)^p \sigma_i(E_Q)^{p/p_i}
\end{aligned}$$

$$\le \prod_{i=1}^m \left(\sum_{Q\in\mathcal{S}}\left(\frac{1}{\sigma_i(Q)}\int_Q |f_i|\sigma_i\right)^{p_i}\sigma_i(E_Q)\right)^{p/p_i}$$
$$\le \prod_{i=1}^m \|M_{\sigma_i}^{\mathscr{D}}(f_i)\|_{L^{p_i}(\sigma_i)}^p$$
$$\lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^p.$$

Now consider the case $p\ge \max_i p_i'$. It is sufficient to prove that

$$\|A_{\mathscr{D},\mathcal{S}}(\vec{f\sigma})\|_{L^p(v_{\vec{w}})}\lesssim [\vec{w}]_{A_{\vec{P}}}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}.$$

By duality, it suffices to estimate the $(m+1)$-linear form

$$\int_{\mathbb{R}^n} A_{\mathscr{D},\mathcal{S}}(\vec{f\sigma})gv_{\vec{w}}=\sum_{Q\in\mathcal{S}}\int_Q gv_{\vec{w}}\cdot\prod_{i=1}^m\frac{1}{|Q|}\int_Q f_i\sigma_i$$

for $g\ge 0$ belonging to $L^{p'}(v_{\vec{w}})$. We have

$$\begin{aligned}\sum_{Q\in\mathcal{S}}\int_Q gv_{\vec{w}} &\cdot \prod_{i=1}^m\frac{1}{|Q|}\int_Q f_i\sigma_i\\
&= \sum_{Q\in\mathcal{S}}\frac{v_{\vec{w}}(Q)\prod_{i=1}^m\sigma_i(Q)^{p/p_i'}}{|Q|^{mp}}\cdot\frac{|Q|^{m(p-1)}}{v_{\vec{w}}(Q)\prod_{i=1}^m\sigma_i(Q)^{p/p_i'}}\\
&\quad\cdot\int_Q gv_{\vec{w}}\cdot\prod_{i=1}^m\int_Q f_i\sigma_i\\
&\le [\vec{w}]_{A_{\vec{P}}}\sum_{Q\in\mathcal{S}}\frac{|Q|^{m(p-1)}}{v_{\vec{w}}(Q)\prod_{i=1}^m\sigma_i(Q)^{p/p_i'}}\cdot\int_Q gv_{\vec{w}}\cdot\prod_{i=1}^m\int_Q f_i\sigma_i\\
&\le [\vec{w}]_{A_{\vec{P}}}\sum_{Q\in\mathcal{S}}\frac{2^{m(p-1)}|E_Q|^{m(p-1)}}{v_{\vec{w}}(Q)\prod_{i=1}^m\sigma_i(Q)^{p/p_i'}}\cdot\int_Q gv_{\vec{w}}\cdot\prod_{i=1}^m\int_Q f_i\sigma_i.\end{aligned}$$

By (8.3),

$$|E_Q|\le v_{\vec{w}}(E_Q)^{\frac{1}{mp}}\sigma_1(E_Q)^{\frac{1}{mp_1'}}\cdots\sigma_m(E_Q)^{\frac{1}{mp_m'}}. \tag{8.4}$$

Since $p\ge\max_i\{p_i'\}$ and $E_Q\subset Q$, we have $\sigma_i(Q)^{1-\frac{p}{p_i'}}\le\sigma_i(E_Q)^{1-\frac{p}{p_i'}}$ for any $i=1,\cdots,m$. Therefore,

$$\begin{aligned}&\sum_{Q\in\mathcal{S}}\int_Q gv_{\vec{w}}\cdot\prod_{i=1}^m\frac{1}{|Q|}\int_Q f_i\sigma_i\\
&\qquad\le 2^{m(p-1)}[\vec{w}]_{A_{\vec{P}}}\sum_{Q\in\mathcal{S}}v_{\vec{w}}(E_Q)^{\frac{1}{p'}}\prod_{i=1}^m\sigma_i(E_Q)^{\frac{p-1}{p_i'}}\sigma_i(Q)^{1-\frac{p}{p_i'}}\end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{1}{v_{\vec{w}}(Q)} \int_Q g v_{\vec{w}} \cdot \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i \sigma_i \\
\leq\ & 2^{m(p-1)} [\vec{w}]_{A_{\vec{P}}} \sum_{Q \in \mathcal{S}} v_{\vec{w}}(E_Q)^{\frac{1}{p'}} \prod_{i=1}^m \sigma_i(E_Q)^{\frac{1}{p_i}} \frac{1}{v_{\vec{w}}(Q)} \int_Q g v_{\vec{w}} \\
&\cdot \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i \sigma_i \\
\leq\ & 2^{m(p-1)} [\vec{w}]_{A_{\vec{P}}} \left(\sum_{Q \in \mathcal{S}} \left(\frac{1}{v_{\vec{w}}(Q)} \int_Q g v_{\vec{w}} \right)^{p'} v_{\vec{w}}(E_Q) \right)^{1/p'} \\
&\cdot \prod_{i=1}^m \left(\sum_{Q \in \mathcal{S}} \left(\frac{1}{\sigma_i(Q)} \int_Q f_i \sigma_i \right)^{p_i} \sigma_i(E_Q) \right)^{1/p_i} \\
\leq\ & 2^{m(p-1)} [\vec{w}]_{A_{\vec{P}}} \| M^{\mathscr{D}}_{v_{\vec{w}}}(g) \|_{L^{p'}(v_{\vec{w}})} \prod_{i=1}^m \| M^{\mathscr{D}}_{\sigma_i}(f_i) \|_{L^{p_i}(\sigma_i)} \\
\lesssim\ & 2^{m(p-1)} [\vec{w}]_{A_{\vec{P}}} \| g \|_{L^{p'}(v_{\vec{w}})} \prod_{i=1}^m \| f_i \|_{L^{p_i}(\sigma_i)},
\end{aligned}
$$

where we have used the boundedness of the weighted maximal function in the last step. For the other cases we use duality. Notice that the operator $A_{\mathscr{D},\mathcal{S}}$ is self adjoint as a multilinear operator, in the sense that for any i, $i = 1, \dots, m$, we have

$$
\int_{\mathbb{R}^n} A_{\mathscr{D},\mathcal{S}}(f_1, \dots, f_m) g = \int_{\mathbb{R}^n} A_{\mathscr{D},\mathcal{S}}(f_1, \dots, f_{i-1}, g, f_{i+1}, \dots f_m) f_i.
$$

Without loss of generality suppose $p_1' \geq \max(p, p_2', \dots, p_m')$. Hence, by duality and self adjointness we have

$$
\begin{aligned}
\| A_{\mathscr{D},\mathcal{S}} \|_{L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(v_{\vec{w}})} &= \| A_{\mathscr{D},\mathcal{S}} \|_{L^{p'}(v_{\vec{w}}^{1-p'}) \times \cdots \times L^{p_m}(w_m) \to L^{p_1'}(w_1^{1-p_1'})} \\
&\lesssim [\vec{w}^1]_{\vec{P}^1} = [\vec{w}]_{A_{\vec{P}}}^{\frac{p_1'}{p}}.
\end{aligned}
$$

□

9. Recent advances on the control of multilinear Calderón–Zygmund operators

Below is a partial list of important contributions to find the sharp bounds for multilinear Calderón–Zygmund operators.

- Control in norm from above by sparse operators of classical m-CZOs using the local mean oscillation formula and generalization of the A_2 theorem (W.D., A.K. Lerner and C. Pérez, [15]).
- Sharp bounds for sparse operators in the general case avoiding the use of extrapolation and A_p theorem for m-CZOs for the case $1 < p < \infty$ (K.L., K. Moen and

W. Sun [38]). The case when $1/m < p < 1$ was still open since their result relied on the domination theorem in [15] which only holds for Banach function spaces.
- Pointwise control of log-Dini continuous m-CZOs by sparse operators (A.K. Lerner and F. Nazarov [36] and J.M. Conde-Alonso and G. Rey [12]).
- Pointwise control of Dini continuous CZOs by sparse operators (M.T. Lacey [30]); tracking the precise constants (T. Hytönen, L. Roncal and O. Tapiola [26]); and further simplifications of the proof (A.K. Lerner [34]).
- Pointwise control of Dini continuous m-CZOs by sparse operators taking care of the precise constants and applications to several multilinear operators (W.D., M. Hormozi and K.L. [14]).

10. Domination theorem for multilinear Calderón-Zygmund operators

In Section 2 we have introduced the standard multilinear Calderón-Zygmund operators. Now we shall relax the kernel estimates slightly. We say that T is a *ω-bilinear Calderón–Zygmund operator* if it is a bilinear operator originally defined on the product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \tag{10.1}$$

and for some $1 \leq q_1, q_2 < \infty$ it extends to a bounded bilinear operator from $L^{q_1} \times L^{q_2}$ to L^q, where $1/q_1 + 1/q_2 = 1/q$, and if there exists a function K, defined off the diagonal $x = y = z$ in $(\mathbb{R}^n)^3$, satisfying

$$T(f_1, f_2)(x) = \iint_{(\mathbb{R}^n)^2} K(x, y, z) f_1(y) f_2(z) dydz, \tag{10.2}$$

for all $x \notin \operatorname{supp} f_1 \cap \operatorname{supp} f_2$. The kernel K must also satisfy, for some constants $C_K > 0$ and $\tau \in (0, 1)$, the following size condition

$$|K(x, y, z)| \leq \frac{C_K}{(|x-y| + |x-z|)^{2n}}, \tag{10.3}$$

and, the smoothness estimate

$$\begin{aligned}|K(x+h, y, z) - K(x, y, z)| &+ |K(x, y+h, z) - K(x, y, z)| \\ &+ |K(x, y, z+h) - K(x, y, z)| \\ &\leq \frac{1}{(|x-y| + |x-z|)^{2n}} \omega\left(\frac{|h|}{|x-y| + |x-z|}\right),\end{aligned}$$

whenever $|h| \leq \tau \max(|x-y|, |x-z|)$.
If $\omega : [0, \infty) \to [0, \infty)$ is a modulus of continuity (i.e. it is increasing, subadditive ($\omega(t + s) \leq \omega(t) + \omega(s)$) and $\omega(0) = 0$), the kernel K is said to be a log-Dini-continuous kernel if ω satisfies the following condition

$$||\omega||_{\text{log-Dini}} := \int_0^1 \omega(t) \left(1 + \log\left(\frac{1}{t}\right)\right) \frac{dt}{t} < \infty. \tag{10.4}$$

We are mostly interested in the weaker case when K is a *Dini*(a)-conti-nuous kernel. Namely, when ω satisfies the following condition:

$$||\omega||_{\text{Dini(a)}} := \int_0^1 \omega^a(t) \frac{dt}{t} < \infty. \tag{10.5}$$

In the case $a=1$, we will denote $||\omega||_{\mathrm{Dini(a)}}$ simply as $||\omega||_{\mathrm{Dini}}$.
Given a bilinear Calderón-Zygmund operator T, the maximal truncation of T is defined as the operator $T_\sharp$ given by

$$T_\sharp(f_1,f_2)(x)=\sup_{\varepsilon>0}|T_\varepsilon(f_1,f_2)(x)|, \tag{10.6}$$

where T_ε is the ε-truncation of T

$$T_\varepsilon(f_1,f_2)(x)=\int_{|x-y|^2+|x-z|^2>\varepsilon^2}K(x,y,z)f_1(y)f_2(z)dydz. \tag{10.7}$$

Our goal in this section is showing that, for bilinear CZO T, whose kernel satisfies the Dini(1) condition, then T can be controlled by a finite summation of sparse operators introduced in Section 6. Recall that the dyadic systems are defined by

$$\mathcal{D}^u:=\{2^{-k}([0,1)^u+m+(-1)^k\tfrac{1}{3}u):k\in\mathbb{Z},m\in\mathbb{Z}^n\},\quad u\in\{0,1,2\}^n. \tag{10.8}$$

Our main result in this section states as follows

Theorem 10.1. *Let T be a bilinear ω-Calderón–Zygmund operator with ω satisfying the Dini condition. Then for any pair of compactly supported functions $f_1,f_2\in L^1(\mathbb{R}^n)$, there exist sparse collections $\mathcal{S}^u\subset\mathcal{D}^u$, $u=1,2,\ldots,3^n$, such that*

$$T(f_1,f_2)(x)\le c_n(\|T\|_{L^{q_1}\times L^{q_2}\to L^q}+C_K+\|\omega\|_{\mathrm{Dini}})\sum_{u=1}^{3^n}\mathcal{A}_{\mathcal{S}^u}(f_1,f_2)(x), \tag{10.9}$$

for almost every $x\in\mathbb{R}^n$, where the constant c_n depends only on the dimension and $\|T\|_{L^{q_1}\times L^{q_2}\to L^q}$ denotes the norm of the operator.

Theorem 10.1 has been proved in [14] (actually for $T_\sharp$) using a similar arguments with [26]. However, in this lecture note, we shall introduce a new proof follows from Lerner's recent idea [34]. With Hänninen's arguments [23] in hand, it suffices to prove the following

Theorem 10.2. *Let T be a bilinear ω-Calderón–Zygmund operator with ω satisfying the Dini condition. Then for any pair of compactly supported functions $f_1,f_2\in L^1(\mathbb{R}^n)$, there exists a sparse collection $\mathcal{S}$, such that for a.e. $x\in\mathbb{R}^n$*

$$T(f_1,f_2)(x)\le c_n(\|T\|_{L^{q_1}\times L^{q_2}\to L^q}+C_K+\|\omega\|_{\mathrm{Dini}})\sum_{Q\in\mathcal{S}}\langle f_1\rangle_{3Q}\langle f_2\rangle_{3Q}\mathbf{1}_Q(x). \tag{10.10}$$

As that in [34], we define the bilinear grand maximal truncated operator $\mathcal{M}_T$ by

$$\mathcal{M}_T(f_1,f_2)(x):=\sup_{Q\ni x}\operatorname*{ess\,sup}_{\xi\in Q}\left(|T(f_1,f_2)(\xi)-T(f_1\chi_{3Q},f_2\chi_{3Q})(\xi)|\right).$$

Given a cube Q_0, for $x\in Q_0$ we define a local version of $\mathcal{M}_T$ by

$$\mathcal{M}_{T,Q_0}(f_1,f_2)(x):=\sup_{Q\ni x,Q\subset Q_0}\operatorname*{ess\,sup}_{\xi\in Q}\left(|T(f_1\chi_{3Q_0},f_2\chi_{3Q_0})(\xi)-T(f_1\chi_{3Q},f_2\chi_{3Q})(\xi)|\right).$$

We have the following lemma.

Lemma 10.3. *The following pointwise estimate holds*

(1) *for a.e. $x\in Q_0$,*

$$T(f_1\chi_{3Q_0},f_2\chi_{3Q_0})(x)\le c_n\|T\|_{L^1\times L^1\to L^{\frac{1}{2},\infty}}|f_1(x)f_2(x)|+\mathcal{M}_{T,Q_0}(f_1,f_2)(x);$$

(2) *for all* $x \in \mathbb{R}^n$,

$$\mathcal{M}_T(f_1, f_2)(x) \le c_n(\|\omega\|_{\text{Dini}} + C_K)\mathcal{M}(f_1, f_2)(x) + T_\sharp(f_1, f_2)(x).$$

Proof. Suppose that $x \in Q^\circ$ and let x be a point of approximate continuity of $T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})$. Then for every $\varepsilon > 0$, the sets

$$E_s(x) := \{y \in B(x,s) : |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(y) - T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)| < \varepsilon\}$$

satisfy $\lim_{s\to 0} \frac{|E_s(x)|}{|B(x,s)|} = 1$. Denote by $Q(x,s)$ the smallest cube centered at x and containing $B(x,s)$. Let $s > 0$ be so small that $Q(x,s) \subset Q_0$. Then for a.e. $y \in E_s(x)$,

$$\begin{aligned} T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) &< T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(y) + \varepsilon \\ &\le T(f_1\chi_{3Q(x,s)}, f_2\chi_{3Q(x,s)})(y) + \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \varepsilon. \end{aligned}$$

It follows that

$$\begin{aligned} T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) &\le \operatorname*{ess\,inf}_{y\in E_s(x)} T(f_1\chi_{3Q(x,s)}, f_2\chi_{3Q(x,s)})(y) + \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \varepsilon \\ &\le \left(\frac{1}{|E_s(x)|}\int_{E_s(x)} |T(f_1\chi_{3Q(x,s)}, f_2\chi_{3Q(x,s)})(y)|^\delta dy\right)^{\frac{1}{\delta}} \\ &\quad + \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \varepsilon \\ &\le \|T(f_1\chi_{3Q(x,s)}, f_2\chi_{3Q(x,s)})\|_{L^{\frac{1}{2},\infty}(E_s(x), \frac{dx}{|E_s(x)|})} \\ &\quad + \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \varepsilon \\ &\le \|T\|_{L^1\times L^1\to L^{\frac{1}{2},\infty}} \frac{1}{|E_s(x)|^2}\int_{3Q(x,s)} |f_1(y)|dy \int_{3Q(x,s)} |f_2(y)|dy \\ &\quad + \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \varepsilon. \end{aligned}$$

Assuming additionally that x is a Lebesgue point of f_1 and f_2 and letting subsequently $s \to 0$ and $\varepsilon \to 0$, we obtain (i).
Now we turn to prove (ii). Let $x, \xi \in Q$. Denote by B_x the closed ball centered at x of radius $2\text{diam}Q$, then $3Q \subset B_x$. Set

$$\widetilde{T}_\varepsilon(f_1, f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} K(x,y,z)f_1(y)f_2(z)dydz.$$

We have

$$\begin{aligned} &|T(f_1, f_2)(\xi) - T(f_1\chi_{3Q}, f_2\chi_{3Q})(\xi)| \\ &= |T(f_1, f_2)(\xi) - T(f_1\chi_{3Q}, f_2\chi_{3Q})(\xi) - \widetilde{T}_{2\text{diam}Q}(f_1, f_2)(\xi)| \\ &+ |\widetilde{T}_{2\text{diam}Q}(f_1, f_2)(\xi) - T_{2\text{diam}Q}(f_1, f_2)(\xi)| \\ &\quad + |T_{2\text{diam}Q}(f_1, f_2)(\xi) - T_{2\text{diam}Q}(f_1, f_2)(x)| + |T_{2\text{diam}Q}(f_1, f_2)(\xi)| \\ &:= I + II + III + IV. \end{aligned}$$

By size condition,

$$I = |T(f_1\chi_{B_x\setminus 3Q}, f_2\chi_{B_x}) + T(f_1\chi_{3Q}, f_1\chi_{B_x\setminus 3Q})| \le c_n C_K \mathcal{M}(f_1, f_2)(x);$$

$$II = \left| \int_{\substack{\max\{|x-y|,|x-z|\}\le 2\mathrm{diam}Q \\ |x-y|^2+|x-z|^2>(2\mathrm{diam}Q)^2}} K(x,y,z)f_1(y)f_2(z)dydz \right| \le c_n C_K \mathcal{M}(f_1,f_2)(x).$$

By definition, $IV \le T_\sharp(f_1,f_2)(x)$. Finally, by smoothness condition,

$$\begin{aligned} III &\le \int_{|x-y|^2+|x-z|^2>(2\mathrm{diam}Q)^2} |K(x,y,z)-K(\xi,y,z)|\cdot|f_1(y)|\cdot|f_2(z)|dydz \\ &= \sum_{k=1}^{\infty} \int_{(2^k\mathrm{diam}Q)^2<|x-y|^2+|x-z|^2\le(2^{k+1}\mathrm{diam}Q)^2} |K(x,y,z)-K(\xi,y,z)|\cdot|f_1(y)|\cdot|f_2(z)|dydz \\ &\le \sum_{k=1}^{\infty} \frac{\omega(2^{-k})}{(2^k\mathrm{diam}Q)^{2n}} \int_{B(x,2^{k+1}\mathrm{diam}Q)} |f_1(y)|dy \int_{B(x,2^{k+1}\mathrm{diam}Q)} |f_2(z)|dz \\ &\le c_n \mathcal{M}(f_1,f_2)(x) \sum_{k=1}^{\infty} \omega(2^{-k}) \\ &\le c_n \|\omega\|_{\mathrm{Dini}} \mathcal{M}(f_1,f_2)(x). \end{aligned}$$

□

Now we are ready to prove Theorem 10.2. Denote

$$C_T := \|T\|_{L^1\times L^1\to L^{\frac12,\infty}} + \|T_\sharp\|_{L^1\times L^1\to L^{\frac12,\infty}} + \|\omega\|_{\mathrm{Dini}} + C_K$$

.

Proof of Theorem 10.2. Fix a cube $Q_0 \subset \mathbb{R}^n$. We shall prove the following recursive inequality,

$$|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0} \le c_n C_T \langle f\rangle_{3Q_0} + \sum_j |T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j}, \tag{10.11}$$

where P_j are disjoint dyadic subcubes of Q_0, say $\mathcal{D}(Q_0)$ and moreover, $\sum_j |P_j| \le \frac12|Q_0|$. Once (10.11) is verified, then Theorem 10.2 follows immediately.
Next, observe that for arbitrary pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,

$$\begin{aligned} &|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0} \\ &= |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{P_j} \\ &\le |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j} \\ &+ \sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j} \end{aligned}$$

Hence, in order to prove the recursive claim, it suffices to show that one can select pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ with $\sum_j |P_j| \le \frac12|Q_0|$ and such that for a.e. $x \in Q_0$,

$$\begin{aligned} &\sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j} \\ &\quad + |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j} \le c_n C_T \langle f\rangle_{3Q_0}. \end{aligned}$$

By Lemma 10.3 we have $\|\mathcal{M}_T\|_{L^1\times L^1\to L^{\frac{1}{2},\infty}} \le \alpha_n C_T$. Therefore, one can choose c_n such that the set

$$\begin{aligned} E :=&\{x \in Q_0 : |f_1(x)f_2(x)| > c_n\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}\} \\ &\cup \{x \in Q_0 : \mathcal{M}_{T,Q_0}(f_1,f_2)(x) > c_nC_T\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}\} \end{aligned}$$

will satisfy $|E| \le \frac{1}{2^{n+2}}|Q_0|$. The Calderón-Zygmund decomposition applied to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$ produces pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \le |P_j \cap E| < \frac{1}{2}|P_j|$$

and $|E \setminus \cup_j P_j| = 0$. It follows that $\sum_j |P_j| \le \frac{1}{2}|Q_0|$ and $P_j \cap E^c \ne \emptyset$. Therefore,

$$\operatorname*{ess\,sup}_{\xi\in P_j} |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)| \le c_nC_T\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}.$$

On the other hand, by Lemma 10.3, for a.e. $x \in Q_0 \setminus \cup_j P_j$, we have

$$|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)| \le c_nC_T\langle|f_1|\rangle_{3Q_0}\langle|f_2|\rangle_{3Q_0}.$$

Therefore, combining the estimates we arrive at (10.11).
Now with (10.11), suppose that $\operatorname{supp} f_1 \cup \operatorname{supp} f_2 \subset Q_0 \in \mathcal{D}^{u_0}$ for some $u_0 \in \{0,1,2\}^n$. Without loss of generality we can assume that $u_0 = 0$ and $Q_0 = [0,1)^n$. Then we construct a partition of $\mathbb{R}^n$ in the following way, which is slight different with that in Lerner's paper [34]. Denote $\operatorname{bro}(Q_0) := \{Q \subset \widehat{Q_0} : \ell(Q) = \ell(Q_0), Q \ne Q_0\}$, where $\widehat{Q_0}$ is the dyadic parent of Q_0. Denote

$$\mathcal{P}(Q_0) := \{Q_0\} \cup \bigcup_{k=0}^{\infty} \operatorname{bro}(Q_0^{(k)}),$$

where $Q^{(k)}$ denotes the k-th ancestor of Q. Then $\mathcal{P}(Q_0)$ is a partition of the quadrant which contains Q_0. Let Q_i, $i = 1, \cdots, 2^n-1$ be the mirroring of Q_0 in the other quadrants. Then our partition of $\mathbb{R}^n$ is

$$\mathcal{P} = \bigcup_{i=0}^{2^n-1} \mathcal{P}(Q_i).$$

It is easy to check that, for any $P \in \mathcal{P}$, $Q_0 \subset 3P$. Then apply (10.11) to each $P \in \mathcal{P}$, we obtain

$$\begin{aligned} |T(f_1,f_2)(x)| &= \sum_{P\in\mathcal{P}} |T(f_1,f_2)(x)|\chi_P \\ &= \sum_{P\in\mathcal{P}} |T(f_1\chi_{3P}, f_2\chi_{3P})(x)|\chi_P \\ &\le c_nC_T \sum_{P\in\mathcal{P}} \sum_{Q\in\mathcal{S}_P} \langle f_1\rangle_{3Q}\langle f_2\rangle_{3Q}\mathbf{1}_Q(x). \end{aligned}$$

This completes the proof of Theorem 10.2. □

11. A_p-A_∞ BOUND OF BILINEAR SPARSE OPERATORS

In this section, we study the A_p-A_∞ bound of bilinear Calderón-Zygmund operators. With the domination theorem in hand, it suffices to study the corresponding estimates for bilinear sparse operators. Indeed, we shall study a more general class of sparse operators. To be precise, we consider the following type of sparse operators

$$\mathcal{A}_{p_0,\gamma,\mathcal{S}}(\vec{f})(x) := \left(\sum_{Q\in S}\left[\prod_{i=1}^{2}\langle f_i\rangle_{Q,p_0}\right]^{\gamma}\mathbf{1}_Q(x)\right)^{1/\gamma},$$

where for any cube Q,

$$\langle f\rangle_{Q,p_0} := \left(\frac{1}{|Q|}\int_Q |f(x)|^{p_0}dx\right)^{\frac{1}{p_0}}.$$

Our main result can be stated as follows.

Theorem 11.1. *Let $\gamma > 0$. Suppose that $p_0 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let w and $\vec{\sigma}$ be weights satisfying that $[w,\vec{\sigma}]_{A_{\vec{P}/p_0}} < \infty$ and $w, \sigma_i \in A_\infty$ for $i = 1, 2$. If $\gamma \geq p_0$, then*

$$\begin{aligned}&\|\mathcal{A}_{p_0,\gamma,\mathcal{S}}(\cdot\sigma_1,\cdot\sigma_2)\|_{L^{p_1}(\sigma_1)\times L^{p_2}(\sigma_2)\to L^p(w)}\\ &\quad\lesssim [w,\vec{\sigma}]^{\frac{1}{p}}_{A_{\vec{P}/p_0}}\Big(\prod_{i=1}^{2}[\sigma_i]^{\frac{1}{p_i}}_{A_\infty} + [w]^{(\frac{1}{\gamma}-\frac{1}{p})_+}_{A_\infty}\sum_{j=1}^{m}\prod_{i\neq j}[\sigma_i]^{\frac{1}{p_i}}_{A_\infty}\Big),\end{aligned}$$

where

$$\left(\frac{1}{\gamma}-\frac{1}{p}\right)_+ := \max\left\{\frac{1}{\gamma}-\frac{1}{p}, 0\right\}.$$

If $\gamma < p_0$, then the above result still holds for all $p > \gamma$.

Remark 11.2. If $p_0 = \gamma = 1$, then by the domination theorem, the A_p-A_∞ bound for bilinear Calderón-Zygmund operators follows immediately (see [40]). Indeed, the above result actually provided the A_p-A_∞ bound for a large class of operators. For example, one can also use it to bound the bilinear square functions and bilinear Fourier multipliers.

To prove Theorem 11.1, we need the following formula.

Proposition 11.3. *Let $1 < s < \infty$, σ be a positive Borel measure and*

$$\phi = \sum_{Q\in\mathcal{D}}\alpha_Q\mathbf{1}_Q, \qquad \phi_Q = \sum_{Q'\subset Q}\alpha_{Q'}\mathbf{1}_{Q'}.$$

Then

$$\|\phi\|_{L^s(\sigma)} \leq C_s\Big(\sum_{Q\in\mathcal{D}}\alpha_Q(\langle\phi_Q\rangle^{\sigma}_{Q})^{s-1}\sigma(Q)\Big)^{1/s}.$$

Indeed, the reverse inequality also holds. See [8, 52] for details. However, to prove Theorem 11.1, Proposition 11.3 suffices.

Proof. We use the following elementary inequality

$$\left(\sum_i a_i\right)^s \leq s\sum_i a_i\left(\sum_{j\geq i}a_j\right)^{s-1}, \tag{11.1}$$

where $\{a_i\}_{i\in\mathbb{Z}}$ is a sequence of non-negative summable numbers. To see (11.1), notice that

$$b^s - a^s \le s(b-a)b^{s-1}, \text{ for } 0 \le a \le b.$$

Then we have, for any j,

$$\left(\sum_{i\le j} a_i\right)^s - \left(\sum_{i<j} a_i\right)^s \le s a_j \left(\sum_{i\le j} a_i\right)^{s-1}$$

Then sum over j, we arrive at (11.1). Now consider the case $1 < s \le 2$ first. We have

$$\begin{aligned}
\|\phi\|_{L^s(\sigma)}^s &= \int \left(\sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q\right)^s d\sigma \\
&\le s \int \sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q \left(\sum_{Q'\subset Q} \alpha_{Q'} \mathbf{1}_{Q'}\right)^{s-1} d\sigma \\
&= s \sum_{Q\in\mathcal{D}} \alpha_Q \int_Q \phi_Q^{s-1} d\sigma \\
&\le s \sum_{Q\in\mathcal{D}} \alpha_Q \sigma(Q)(\langle \phi_Q \rangle_Q^\sigma)^{s-1}.
\end{aligned}$$

It remains to study the case $s > 2$. Denote $k = \lceil s-2 \rceil$, then apply (11.1) k times we get

$$\begin{aligned}
\|\phi\|_{L^s(\sigma)}^s &= \int \left(\sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q\right)^s d\sigma \\
&\le s \int \sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q \left(\sum_{Q_1\subset Q} \alpha_{Q_1} \mathbf{1}_{Q_1}\right)^{s-1} d\sigma \\
&\le s(s-1)\cdots(s-k) \int \sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q \sum_{Q_1\subset Q} \alpha_{Q_1} \mathbf{1}_{Q_1} \cdots \left(\sum_{Q_{k+1}\subset Q_k} \alpha_{Q_{k+1}} \mathbf{1}_{Q_{k+1}}\right)^{s-k-1} d\sigma \\
&= c(s) \sum_{Q\in\mathcal{D}} \alpha_Q \sum_{Q_1\subset Q} \alpha_{Q_1} \cdots \sum_{Q_k\subset Q_{k-1}} \alpha_{Q_k} \int \left(\sum_{Q_{k+1}\subset Q_k} \alpha_{Q_{k+1}} \mathbf{1}_{Q_{k+1}}\right)^{s-k-1} d\sigma \\
&\le c(s) \sum_{Q\in\mathcal{D}} \alpha_Q \sum_{Q_1\subset Q} \alpha_{Q_1} \cdots \sum_{Q_k\subset Q_{k-1}} \alpha_{Q_k} (\langle \phi_{Q_k} \rangle_{Q_k}^\sigma)^{s-k-1} \sigma(Q_k) \\
&= c(s) \int \sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q \sum_{Q_1\subset Q} \alpha_{Q_1} \mathbf{1}_{Q_1} \cdots \sum_{Q_k\subset Q_{k-1}} \alpha_{Q_k} (\langle \phi_{Q_k} \rangle_{Q_k}^\sigma)^{s-k-1} \mathbf{1}_{Q_k} d\sigma \\
&\le c(s) \int \Big(\sum_{Q\in\mathcal{D}} \alpha_Q \mathbf{1}_Q\Big)^k \Big(\sum_{Q\in\mathcal{D}} \alpha_Q (\langle \phi_Q \rangle_Q^\sigma)^{s-k-1} \mathbf{1}_Q\Big) d\sigma
\end{aligned}$$

$$\leq c(s)\int \Big(\sum_{Q\in\mathcal{D}}\alpha_Q\mathbf{1}_Q\Big)^{k+\frac{k}{s-1}}\Big(\sum_{Q\in\mathcal{D}}\alpha_Q(\langle\phi_Q\rangle_Q^\sigma)^{s-1}\mathbf{1}_Q\Big)^{\frac{s-k-1}{s-1}}d\sigma$$
$$\leq c(s)\|\phi\|_{L^s(\sigma)}^{\frac{sk}{s-1}}\Big(\sum_{Q\in\mathcal{D}}\alpha_Q(\langle\phi_Q\rangle_Q^\sigma)^{s-1}\sigma(Q)\Big)^{\frac{s-k-1}{s-1}},$$

where $c(s) = s(s-1)\cdots(s-k)$. Then the desired estimates follows provided that $\|\phi\|_{L^s(\sigma)} < \infty$. □

We also need the following proposition.

Proposition 11.4. *Let $\mathcal{S}$ be a sparse family and $0 \leq \gamma, \eta < 1$ satisfying $\gamma + \eta < 1$. Then*

$$\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\langle u\rangle_Q^\gamma\langle v\rangle_Q^\eta|Q| \lesssim \langle u\rangle_R^\gamma\langle v\rangle_R^\eta|R|. \tag{11.2}$$

Proof. Indeed, set $1/r := \gamma+\eta$, $1/s := \gamma + (1-1/r)/2$ and $1/s' := 1-1/s$. By sparseness and Kolmogorov's inequality, we have

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\langle u\rangle_Q^\gamma\langle v\rangle_Q^\eta|Q| &\leq 2\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\langle u\rangle_Q^\gamma\langle v\rangle_Q^\eta|E_Q|\\ &\leq 2\int_R M(u\mathbf{1}_R)^\gamma M(v\mathbf{1}_R)^\eta dx\\ &\leq 2\Big(\int_R M(u\mathbf{1}_R)^{s\gamma}\Big)^{1/s}\Big(\int_R M(v\mathbf{1}_R)^{s'\eta}\Big)^{1/s'}\\ &\lesssim \langle u\rangle_R^\gamma|R|^{1/s}\langle v\rangle_R^\eta|R|^{1/s'} = \langle u\rangle_R^\gamma\langle v\rangle_R^\eta|R|.\end{aligned}$$

□

To prove Theorem 11.1, we make the following two observations.

Observation 1. Our first observation is that we can reduce the problem to study the case of $p_0 = 1$. Indeed, consider the two weight norm inequality

$$\|\mathcal{A}_{p_0,\gamma,\mathcal{S}}(f_1,f_2)\|_{L^p(w)} \leq \mathcal{N}(\vec{P},p_0,\gamma,w,\vec{\sigma})\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}, \tag{11.3}$$

where we use $\mathcal{N}(\vec{P},p_0,\gamma,w,\vec{\sigma})$ to denote the best constant such that (11.3) holds. Rewrite (11.3) as

$$\|\mathcal{A}_{p_0,\gamma,\mathcal{S}}(f_1^{\frac{1}{p_0}},f_2^{\frac{1}{p_0}})\|_{L^p(w)}^{p_0} \leq \mathcal{N}(\vec{P},p_0,\gamma,w,\vec{\sigma})^{p_0}\|f_1^{\frac{1}{p_0}}\|_{L^{p_1}(w_1)}^{p_0}\|f_2^{\frac{1}{p_0}}\|_{L^{p_2}(w_2)}^{p_0},$$

which is equivalent to the following

$$\|\mathcal{A}_{1,\frac{\gamma}{p_0},\mathcal{S}}(f_1,f_2)\|_{L^{p/p_0}(w)} \leq \mathcal{N}(\vec{P},p_0,\gamma,w,\vec{\sigma})^{p_0}\|f_1\|_{L^{p_1/p_0}(w_1)}\|f_2\|_{L^{p_2/p_0}(w_2)}.$$

Therefore, if we denote by $\mathcal{N}(\vec{P},\gamma,w,\sigma)$ the best constant for the case $p_0 = 1$, then the best constant for general p_0 would be $\mathcal{N}(\vec{P}/p_0,\gamma/p_0,w,\sigma)^{1/p_0}$. Therefore, it suffices to study the case of $p_0 = 1$.

Our second observation is the following

Observation 2. Consider the case $p > \gamma$. Let $\mathcal{N}$ denote the best constant such that the following inequality holds

$$\|\mathcal{A}_{1,\gamma,\mathcal{S}}(f_1\sigma_1, f_2\sigma_2)\|_{L^p(w)} \le \mathcal{N}\|f_1\|_{L^{p_1}(\sigma_1)}\|f_2\|_{L^{p_2}(\sigma_2)}. \tag{11.4}$$

Then (11.4) is equivalent to the following inequality with $\mathcal{N}' \simeq \mathcal{N}^\gamma$

$$\Big\|\Big(\sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}^\gamma \le \mathcal{N}'\|f_1\|_{L^{\frac{p_1}{\gamma}}(\sigma_1)}\|f_2\|_{L^{\frac{p_2}{\gamma}}(\sigma_2)}. \tag{11.5}$$

Indeed, on one hand, if (11.5) holds, we have

$$\begin{aligned}
&\|\mathcal{A}_{1,\gamma,\mathcal{S}}(f_1\sigma_1, f_2\sigma_2)\|_{L^p(w)}\\
&\qquad\le \Big\|\Big(\sum_{Q\in\mathcal{S}} \langle M_{\mathcal{D}}^{\sigma_1}(f_1)^\gamma\rangle_Q^{\sigma_1}\langle M_{\mathcal{D}}^{\sigma_2}(f_2)^\gamma\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}\\
&\qquad\lesssim \mathcal{N}\|M_{\mathcal{D}}^{\sigma_1}(f_1)^\gamma\|_{L^{p_1/\gamma}(\sigma_1)}^{1/\gamma}\|M_{\mathcal{D}}^{\sigma_2}(f_2)^\gamma\|_{L^{p_2/\gamma}(\sigma_2)}^{1/\gamma}\\
&\qquad\le \mathcal{N}\|f_1\|_{L^{p_1}(\sigma_1)}\|f_2\|_{L^{p_2}(\sigma_2)},
\end{aligned}$$

where $M_{\mathcal{D}}^\sigma$ denotes the dyadic weighted maximal function, namely

$$M_{\mathcal{D}}^\sigma(f) = \sup_{Q\in\mathcal{D}} \frac{1}{\sigma(Q)}\int_Q |f(x)|\sigma dx, \tag{11.6}$$

which is bounded from $L^p(\sigma)$ into itself for every $p > 1$. On the other hand, if (11.4) holds, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}\\
&\le \Big\|\Big(\sum_{Q\in\mathcal{S}} (\langle M_{\gamma,\mathcal{D}}^{\sigma_1}(f_1^{1/\gamma})\rangle_Q^{\sigma_1})^\gamma(\langle M_{\gamma,\mathcal{D}}^{\sigma_2}(f_2^{1/\gamma})\rangle_Q^{\sigma_2})^\gamma\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}\\
&\le \mathcal{N}\|M_{\gamma,\mathcal{D}}^{\sigma_1}(f_1^{1/\gamma})\|_{L^{p_1}(\sigma_1)}\|M_{\gamma,\mathcal{D}}^{\sigma_2}(f_2^{1/\gamma})\|_{L^{p_2}(\sigma_2)}\\
&\lesssim \mathcal{N}\|f_1^{1/\gamma}\|_{L^{p_1}(\sigma_1)}\|f_2^{1/\gamma}\|_{L^{p_2}(\sigma_2)},
\end{aligned}$$

where $M_{\gamma,\mathcal{D}}^\sigma(f) = (M_{\mathcal{D}}^\sigma(f^\gamma))^{1/\gamma}$ and we have used in the last step that $p > \gamma$, which implies $p_1, p_2 > \gamma$ and consequently, the boundedness of the maximal functions.
Now we are ready to prove Theorem 11.1.

Proof of Theorem 11.1. First we consider the case $p > \gamma$. With Observation 2, it suffices to estimate

$$\begin{aligned}
\Big\|\Big(\sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}^\gamma &= \sup_{\|h\|_{L^{q'}(w)}=1}\sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\int_Q hdw\\
&= \sup_{\|h\|_{L^{q'}(w)}=1}\sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q),
\end{aligned}$$

where $q = p/\gamma$. For each $i = 1, 2$, let $\mathcal{F}_i$ be the stopping family starting at Q_0 and defined by the stopping condition

$$\mathrm{ch}_{\mathcal{F}_i}(F_i) := \{F_i' \in \mathcal{D} : F_i' \text{ maximal such that } \langle f_i\rangle_{F_i'}^{\sigma_i} > 2\langle f_i\rangle_{F_i}^{\sigma_i}\}.$$

Each collection $\mathcal{F}_i$ is σ_i-sparse, since

$$\sum_{F_i'\in \mathrm{ch}_{\mathcal{F}_i}(F_i)} \sigma_i(F_i') \leq \frac{1}{2}\frac{\sum_{F_i'\in \mathrm{ch}_{\mathcal{F}_i}(F_i)}\int_{F_i'} f d\sigma}{\int_{F_i} f d\sigma}\sigma_i(F_i) \leq \frac{1}{2}\sigma_i(F_i).$$

The $\mathcal{F}_i$-stopping parent $\pi_{\mathcal{F}_i}(Q)$ of a cube Q is defined by

$$\pi_{\mathcal{F}_i}(Q) := \{F_i \in \mathcal{F}_i : F_i \text{ minimal such that } F_i \supseteq Q\}.$$

By the stopping condition, for every cube Q we have $\langle f_i\rangle_Q^{\sigma_i} \leq 2\langle f_i\rangle_{\pi_{\mathcal{F}_i}(Q)}^{\sigma_i}$. Let $\mathcal{H}$ be the analogue stopping family associated with h and the weight w, verifying the corresponding properties. By rearranging the summation according to the stopping parents and removing the supremum, we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{w}\langle \sigma_1\rangle_Q^{\gamma}\langle \sigma_2\rangle_Q^{\gamma} w(Q)\\
&\qquad = \Bigg(\sum_{F_1\in\mathcal{F}_1}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset F_1}}\sum_{\substack{H\in\mathcal{H}\\ H\subset F_2}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} + \sum_{F_2\in\mathcal{F}_2}\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subset F_2}}\sum_{\substack{H\in\mathcal{H}\\ H\subset F}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}}\\
&\qquad + \sum_{F_1\in\mathcal{F}_1}\sum_{\substack{H\in\mathcal{H}\\ H\subset F_1}}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset H}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} + \sum_{F_2\in\mathcal{F}_2}\sum_{\substack{H\in\mathcal{H}\\ H\subset F_2}}\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subset H}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}}\\
&\qquad + \sum_{H\in\mathcal{H}}\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subset H}}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset F_1}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} + \sum_{H\in\mathcal{H}}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset H}}\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subset F_2}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}}\Bigg)\\
&\qquad \times \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{w}\lambda_Q\\
&\qquad := I + I' + II + II' + III + III',
\end{aligned}$$

where $\pi(Q) = (F_1, F_2, H)$ means that $\pi_{\mathcal{F}_i}(Q) = F_i$, for all $i = 1, 2$ and $\pi_{\mathcal{H}}(Q) = H$ and

$$\lambda_Q := \langle \sigma_1\rangle_Q^{\gamma}\langle \sigma_2\rangle_Q^{\gamma} w(Q).$$

First, we estimate I. We have

$$\begin{aligned}
I &\leq \sum_{F_1\in\mathcal{F}_1}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset F_1}}\sum_{\substack{H\in\mathcal{H}\\ H\subset F_2}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{w}\lambda_Q\\
&\leq 8\sum_{F_1\in\mathcal{F}_1}\langle f_1\rangle_{F_1}^{\sigma_1}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset F_1}}\langle f_2\rangle_{F_2}^{\sigma_2}\sum_{\substack{H\in\mathcal{H}\\ H\subset F_2}}\langle h\rangle_H^{w}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}}\lambda_Q\\
&\lesssim \sum_{F_1\in\mathcal{F}_1}\langle f_1\rangle_{F_1}^{\sigma_1}\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset F_1}}\langle f_2\rangle_{F_2}^{\sigma_2}\int\Bigg(\sup_{\substack{H'\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H')=F_2}}\langle h\rangle_{H'}^{w}\mathbf{1}_{H'}\Bigg)
\end{aligned}$$

$$\times \sum_{\substack{H\in\mathcal{H}\\ H\subset F_2}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{w(Q)}\mathbf{1}_Q \mathrm{d}w$$

$$\le \sum_{F_1\in\mathcal{F}_1} \langle f_1\rangle_{F_1}^{\sigma_1} \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} \langle f_2\rangle_{F_2}^{\sigma_2} \Big\| \sum_{\substack{H\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H)=F_2}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{w(Q)}\mathbf{1}_Q \Big\|_{L^q(w)}$$

$$\times \Big\| \sup_{\substack{H'\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H')=F_2}} \langle h\rangle_{H'}^{w}\mathbf{1}_{H'} \Big\|_{L^{q'}(w)}$$

$$\le \Big(\sum_{F_1\in\mathcal{F}_1} \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_2}(F_2)=F_1}} (\langle f_1\rangle_{F_1}^{\sigma_1}\langle f_2\rangle_{F_2}^{\sigma_2})^q \Big\| \sum_{\substack{H\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H)=F_2}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{w(Q)}\mathbf{1}_Q \Big\|_{L^q(w)}^q \Big)^{1/q}$$

$$\times \Big(\sum_{F_1\in\mathcal{F}_1} \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} \sum_{\substack{H'\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H')=F_2}} (\langle h\rangle_{H'}^{w})^{q'} w(H') \Big)^{1/q'}$$

$$\lesssim \Big(\sum_{F_1\in\mathcal{F}_1} \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} (\langle f_1\rangle_{F_1}^{\sigma_1}\langle f_2\rangle_{F_2}^{\sigma_2})^q \Big\| \sum_{\substack{H\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H)=F_2}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{w(Q)}\mathbf{1}_Q \Big\|_{L^q(w)}^q \Big)^{1/q}.$$

Now it remains to estimate the following testing condition

$$\Big\| \sum_{\substack{H\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H)=F_2}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{w(Q)}\mathbf{1}_Q \Big\|_{L^q(w)}^q = \Big\| \sum_{\substack{H\in\mathcal{H}\\ \pi_{\mathcal{F}_2}(H)=F_2}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q \Big\|_{L^q(w)}^q$$

$$\le \Big\| \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q \Big)^{1/\gamma} \Big\|_{L^p(w)}^p.$$

By the monotonicity of ℓ^γ norm on γ, it suffices to estimate it for small γ. Therefore, without loss of generality, we can assume that $\gamma < 1$ with $(p/\gamma)' < p_1 = \max\{p_1, p_2\}$. Then it is easy to check that

$$0 \le \gamma - \frac{\gamma p_1'}{p_2'} < 1,\ 0 \le 1 - \frac{\gamma p_1'}{p} < 1, \tag{11.7}$$

and

$$\gamma - \frac{\gamma p_1'}{p_2'} + 1 - \frac{\gamma p_1'}{p} < 1. \tag{11.8}$$

By Proposition 11.3, we have

$$\Big\| \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma \mathbf{1}_Q \Big)^{1/\gamma} \Big\|_{L^p(w)} \eqsim \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \lambda_Q \Big(\frac{1}{w(Q)} \sum_{Q'\subset Q} \langle\sigma_1\rangle_{Q'}^\gamma\langle\sigma_2\rangle_{Q'}^\gamma w(Q') \Big)^{\frac{p}{\gamma}-1} \Big)^{\frac{1}{p}}$$

$$\lesssim [w,\vec\sigma]_{A_{\vec P}}^{\frac{(p-\gamma)p_1'}{p^2}}\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\lambda_Q\Big(\frac{1}{w(Q)}\sum_{Q'\subset Q}\langle\sigma_2\rangle_{Q'}^{\gamma(1-\frac{p_1'}{p_2'})}\langle w\rangle_{Q'}^{1-\frac{p_1'\gamma}{p}}|Q'|\Big)^{\frac{p}{\gamma}-1}\Big)^{\frac1p}$$

$$\overset{(11.2)}{\lesssim}[w,\vec\sigma]_{A_{\vec P}}^{\frac{(p-\gamma)p_1'}{p^2}}\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\lambda_Q\Big(\frac{1}{w(Q)}\langle\sigma_2\rangle_{Q}^{\gamma(1-\frac{p_1'}{p_2'})}\langle w\rangle_{Q}^{1-\frac{p_1'\gamma}{p}}|Q|\Big)^{\frac{p}{\gamma}-1}\Big)^{\frac1p}$$

$$=[w,\vec\sigma]_{A_{\vec P}}^{\frac{(p-\gamma)p_1'}{p^2}}\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma+(1-\frac{p_1'}{p_2'})(p-\gamma)}\langle w\rangle_Q^{1-\frac{p_1'(p-\gamma)}{p}}|Q|\Big)^{1/p}$$

$$\lesssim[w,\vec\sigma]_{A_{\vec P}}^{\frac{(p-\gamma)p_1'}{p^2}+\frac1p-\frac{(p-\gamma)p_1'}{p^2}}\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q^{\frac{p}{p_1}}\langle\sigma_2\rangle_Q^{\frac{p}{p_2}}|Q|\Big)^{1/p}$$

$$=[w,\vec\sigma]_{A_{\vec P}}^{\frac1p}\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q^{\frac{p}{p_1}}\langle\sigma_2\rangle_Q^{\frac{p}{p_2}}|Q|\Big)^{1/p}.$$

Then

$$I\lesssim[w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}\Big(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^q\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q^{\frac{p}{p_1}}\langle\sigma_2\rangle_Q^{\frac{p}{p_2}}|Q|\Big)^{1/q}$$

$$\le[w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}\Big(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^q\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q|Q|\Big)^{\frac{p}{p_1}}\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_2\rangle_Q|Q|\Big)^{\frac{p}{p_2}}\Big)^{1/q}$$

$$\le[w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}\Bigg(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\Big(\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^{\frac{p_2}{\gamma}}\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_2\rangle_Q|Q|\Big)^{\frac{p}{p_2}}$$

$$\times\Big(\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q|Q|\Big)^{\frac{p}{p_1}}\Bigg)^{1/q}$$

$$\le[w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}\Big(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^{\frac{p_1}{\gamma}}\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_1\rangle_Q|Q|\Big)^{\frac{\gamma}{p_1}}$$

$$\times\Big(\sum_{F_1\in\mathcal F_1}\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^{\frac{p_2}{\gamma}}\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\langle\sigma_2\rangle_Q|Q|\Big)^{\frac{\gamma}{p_2}}$$

$$\le[w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}[\sigma_1]_{A_\infty}^{\frac{\gamma}{p_1}}[\sigma_2]_{A_\infty}^{\frac{\gamma}{p_2}}\|f_1\|_{L^{p_1/\gamma}(\sigma_1)}\|f_2\|_{L^{p_2/\gamma}(\sigma_2)}.$$

By symmetry, II and III can be reduced to the following testing condition

$$\Big\| \sum_{\substack{F\in\mathcal{F}_2\\ \pi_{\mathcal{H}}(F_2)=H}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{\sigma_2(Q)}\mathbf{1}_Q\Big\|_{L^{(p_2/\gamma)'}(\sigma_2)}^{(p_2/\gamma)'}$$

and

$$\Big\| \sum_{\substack{F\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} \sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2,H)}} \frac{\lambda_Q}{\sigma_2(Q)}\mathbf{1}_Q\Big\|_{L^{(p_2/\gamma)'}(\sigma_2)}^{(p_2/\gamma)'},$$

respectively. It suffices to prove the first one. Now let us consider the case $(p/\gamma)'\geq \max\{p_1,p_2\}$ and $(p/\gamma)'<\max\{p_1,p_2\}$ separately. For the case $(p/\gamma)'<\max\{p_1,p_2\}$, without loss of generality, we may assume that $p_1>p_2$. Again, having into account (11.7) and (11.8) and using Proposition 11.3, we obtain

$$\begin{aligned}
&\Big\| \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^{\gamma-1}\langle w\rangle_Q\mathbf{1}_Q\Big\|_{L^{(\frac{p_2}{\gamma})'}(\sigma_2)}\\
&\qquad\simeq \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \lambda_Q\Big(\frac{1}{\sigma_2(Q)}\sum_{Q'\subset Q}\langle\sigma_1\rangle_{Q'}^\gamma\langle\sigma_2\rangle_{Q'}^\gamma w(Q')\Big)^{(\frac{p_2}{\gamma})'-1}\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}}\\
&\qquad\leq [w,\vec\sigma]_{A_{\vec P}}^{\frac{p_1'\gamma^2}{pp_2}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \lambda_Q\Big(\frac{1}{\sigma_2(Q)}\sum_{Q'\subset Q}\langle\sigma_2\rangle_{Q'}^{\gamma(1-\frac{p_1'}{p_2'})}\langle w\rangle_{Q'}^{1-\frac{\gamma p_1'}{p}}|Q'|\Big)^{(\frac{p_2}{\gamma})'-1}\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}}\\
&\qquad\overset{(11.2)}{\lesssim} [w,\vec\sigma]_{A_{\vec P}}^{\frac{p_1'\gamma^2}{pp_2}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \lambda_Q\Big(\frac{1}{\sigma_2(Q)}\langle\sigma_2\rangle_{Q}^{\gamma(1-\frac{p_1'}{p_2'})}\langle w\rangle_{Q}^{1-\frac{\gamma p_1'}{p}}|Q|\Big)^{(\frac{p_2}{\gamma})'-1}\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}}\\
&\qquad=[w,\vec\sigma]_{A_{\vec P}}^{\frac{p_1'\gamma^2}{pp_2}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^{\gamma(\frac{p_2}{\gamma})'-(\frac{\gamma p_1'}{p_2'}+1)((\frac{p_2}{\gamma})'-1)}\langle w\rangle_Q^{(\frac{p_2}{\gamma})'-\frac{\gamma p_1'}{p}((\frac{p_2}{\gamma})'-1)}|Q|\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}}\\
&\qquad\leq [w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \langle\sigma_1\rangle_Q^{\frac{\gamma(\frac{p_2}{\gamma})'}{p_1}}\langle w\rangle_Q^{(\frac{p_2}{\gamma})'(1-\frac{\gamma}{p})}|Q|\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}},
\end{aligned}$$

where recall $\lambda_Q=\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)$. It remains to consider the case $(p/\gamma)'\geq\max\{p_1,p_2\}$. In this case,

$$\gamma-\frac{p}{p_1'}\geq 0,\ \gamma-\frac{p}{p_2'}\geq 0.$$

Moreover, since we are considering the case $p>\gamma$,

$$\gamma-\frac{p}{p_1'}+\gamma-\frac{p}{p_2'}=2\gamma-2p+1<1.$$

Applying Proposition 11.3 again, we have

$$\begin{aligned}
&\Big\| \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma-1}\langle w\rangle_Q \mathbf{1}_Q\Big\|_{L^{(\frac{p_2}{\gamma})'}(\sigma_2)} \\
&\simeq \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \lambda_Q \Big(\frac{1}{\sigma_2(Q)}\sum_{Q'\subset Q}\langle\sigma_1\rangle_{Q'}^{\gamma}\langle\sigma_2\rangle_{Q'}^{\gamma} w(Q')\Big)^{(\frac{p_2}{\gamma})'-1}\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}} \\
&\le [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p_2}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \lambda_Q \Big(\frac{1}{\sigma_2(Q)}\sum_{Q'\subset Q}\langle\sigma_1\rangle_{Q'}^{\gamma-\frac{p}{p_1'}}\langle\sigma_2\rangle_{Q'}^{\gamma-\frac{p}{p_2'}} |Q'|\Big)^{(\frac{p_2}{\gamma})'-1}\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}} \\
&\overset{(11.2)}{\lesssim} [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p_2}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \lambda_Q \Big(\frac{1}{\sigma_2(Q)}\langle\sigma_1\rangle_{Q}^{\gamma-\frac{p}{p_1'}}\langle\sigma_2\rangle_{Q}^{\gamma-\frac{p}{p_2'}} |Q|\Big)^{(\frac{p_2}{\gamma})'-1}\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}} \\
&= [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p_2}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \langle\sigma_1\rangle_Q^{\frac{\gamma}{p_2-\gamma}(p_2-\frac{p}{p_1'})}\langle\sigma_2\rangle_Q^{\frac{1}{p_2-\gamma}(p_2(1-\gamma)-\frac{p\gamma}{p_2'})}\langle w\rangle_Q |Q|\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}} \\
&\le [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p}} \Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{H}}(Q)=H}} \langle\sigma_1\rangle_Q^{\frac{\gamma(\frac{p_2}{\gamma})'}{p_1}}\langle w\rangle_Q^{(\frac{p_2}{\gamma})'(1-\frac{\gamma}{p})}|Q|\Big)^{\frac{1}{(\frac{p_2}{\gamma})'}}.
\end{aligned}$$

where again $\lambda_Q=\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}w(Q)$. The proof of the case $p>\gamma$ is completed by combining the above estimates. It still remains to consider the case $p\le\gamma$. In this case, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{Q\in\mathcal{S}}\langle f_1\rangle_Q^{\sigma_1}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}^{\gamma} \\
&\lesssim \Big\|\Big(\sum_{F_1\in\mathcal{F}_1}\langle f_1\rangle_{F_1}^{\sigma_1}\sum_{F_2\in\mathcal{F}_2}\langle f_2\rangle_{F_2}^{\sigma_2}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2)}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}^{\gamma} \\
&\le \Big(\sum_{F_1\in\mathcal{F}_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\sum_{F_2\in\mathcal{F}_2}(\langle f_2\rangle_{F_2}^{\sigma_2})^q\Big\|\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=(F_1,F_2)}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\mathbf{1}_Q\Big\|_{L^q(w)}^q\Big)^{\frac{1}{q}} \\
&\lesssim \Big(\sum_{F_1\in\mathcal{F}_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\sum_{\substack{F_2\in\mathcal{F}_2\\ F_2\subset F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^q\Big\|\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\mathbf{1}_Q\Big\|_{L^p(w)}^q\Big)^{\frac{1}{q}} \\
&+ \Big(\sum_{F_2\in\mathcal{F}_2}(\langle f_2\rangle_{F_2}^{\sigma_2})^q\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subset F_2}}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\Big\|\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_1}(Q)=F_1}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\mathbf{1}_Q\Big\|_{L^p(w)}^q\Big)^{\frac{1}{q}}.
\end{aligned}$$

Then by the same calculation as that in the above, we can get the conclusion as desired. □

Theorem 11.5. *Let $\gamma > 0$. Suppose that $p_0 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and set $q = p/\gamma$. Let w and $\vec{\sigma}$ be weights satisfying that $[w, \vec{\sigma}]_{A_{\vec{P}/p_0}} < \infty$. If $\gamma \geq p_0$, then*

$$\begin{aligned}\|\mathcal{A}_{p_0,\gamma,\mathcal{S}}(\cdot\sigma_1, \cdot\sigma_2)\|_{L^{p_1}(\sigma_1)\times L^{p_2}(\sigma_2)\to L^p(w)} \\ \leq [w, \vec{\sigma}]^{1/p}_{A_{\vec{P}/p_0}} ([\vec{\sigma}]^{1/p}_{W^\infty_{\vec{P}}} + \sum_{i=1}^{2} [\vec{\sigma}^i]^{1/\gamma(\frac{p_i}{\gamma})'}_{W^\infty_{\vec{P}i}}),\end{aligned} \tag{11.9}$$

where $[\vec{\sigma}^i]_{W^\infty_{\vec{P}i}} = 1$ if $p \leq \gamma$ and otherwise,

$$\begin{aligned}[\vec{\sigma}^i]_{W^\infty_{\vec{P}i}} = \sup_Q \Big(\int_Q M(\mathbf{1}_Q w)^{\frac{(p_i/\gamma)'}{q'}} \prod_{j\neq i} M(\mathbf{1}_Q \sigma_j)^{\frac{(p_i/\gamma)'}{p_j/\gamma}} dx \Big) \\ \times \Big(\int_Q w^{\frac{(p_i/\gamma)'}{q'}} \prod_{j\neq i} \sigma_j^{\frac{(p_i/\gamma)'}{p_j/\gamma}} dx \Big)^{-1}.\end{aligned}$$

and

$$[\overrightarrow{\sigma}]_{W^\infty_{\vec{P}}} = \sup_Q \Big(\int_Q \prod_{i=1}^m M(\sigma_i \mathbf{1}_Q)^{\frac{p}{p_i}} dx \Big) \Big(\int_Q \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \Big)^{-1} < \infty.$$

If $\gamma < p_0$, then the above result still holds for all $p > \gamma$.

Proof. We can do the same analysis as that in Theorem 11.1. The main difference is, for example, when we estimate I, we have

$$\begin{aligned}I &\lesssim [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} \Big(\sum_{F_1\in\mathcal{F}_1} (\langle f_1\rangle^{\sigma_1}_{F_1})^q \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} (\langle f_2\rangle^{\sigma_2}_{F_2})^q \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \langle\sigma_1\rangle_Q^{\frac{p}{p_1}} \langle\sigma_2\rangle_Q^{\frac{p}{p_2}} |Q| \Big)^{1/q} \\
&\leq [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} \Big(\sum_{F_1\in\mathcal{F}_1} (\langle f_1\rangle^{\sigma_1}_{F_1})^q \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} (\langle f_2\rangle^{\sigma_2}_{F_2})^q \int_{F_2} M(\mathbf{1}_{F_2}\sigma_1)^{\frac{p}{p_1}} M(\mathbf{1}_{F_2}\sigma_2)^{\frac{p}{p_2}} dx \Big)^{1/q} \\
&\leq [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} [\overrightarrow{\sigma}]^{\frac{\gamma}{p}}_{W^\infty_{\vec{P}}} \Big(\sum_{F_1\in\mathcal{F}_1} (\langle f_1\rangle^{\sigma_1}_{F_1})^q \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} (\langle f_2\rangle^{\sigma_2}_{F_2})^q \int_{F_2} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} dx \Big)^{1/q} \\
&\leq [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} [\vec{\sigma}]^{\frac{\gamma}{p}}_{W^\infty_{\vec{P}}} \Big(\int \sum_{F_1\in\mathcal{F}_1} (\langle f_1\rangle^{\sigma_1}_{F_1})^q \mathbf{1}_{F_1} \sum_{\substack{F_2\in\mathcal{F}_2\\ \pi_{\mathcal{F}_1}(F_2)=F_1}} (\langle f_2\rangle^{\sigma_2}_{F_2})^q \mathbf{1}_{F_2} \prod_{i=1}^2 \sigma_i^{p/p_i} dx \Big)^{1/q} \\
&\leq [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} [\vec{\sigma}]^{\frac{\gamma}{p}}_{W^\infty_{\vec{P}}} \Big(\int M^{\sigma_1}_{\mathcal{D}}(f_1)^q M^{\sigma_2}_{\mathcal{D}}(f_2)^q \prod_{i=1}^2 \sigma_i^{p/p_i} dx \Big)^{1/q} \\
&\leq [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} [\vec{\sigma}]^{\frac{\gamma}{p}}_{W^\infty_{\vec{P}}} \|M^{\sigma_1}_{\mathcal{D}}(f_1)\|_{L^{p_1/\gamma}(\sigma_1)} \cdot \|M^{\sigma_2}_{\mathcal{D}}(f_2)\|_{L^{p_2/\gamma}(\sigma_2)} \\
&\lesssim [w, \vec{\sigma}]^{\frac{\gamma}{p}}_{A_{\vec{P}}} [\vec{\sigma}]^{\frac{\gamma}{p}}_{W^\infty_{\vec{P}}} \|f_1\|_{L^{p_1/\gamma}(\sigma_1)} \cdot \|f_2\|_{L^{p_2/\gamma}(\sigma_2)}.\end{aligned}$$

The other terms can be estimated similarly, this completes the proof. □

We also have the following type of bound.

Theorem 11.6. *Let $\gamma > 0$. Suppose that $p_0 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and set $q = p/\gamma$. Let w and $\vec{\sigma}$ be weights satisfying that $[w,\vec{\sigma}]_{A_{\vec{P}/p_0}} < \infty$. If $\gamma \geq p_0$, then*

$$\|\mathcal{A}_{p_0,\gamma,\mathcal{S}}(\cdot\sigma_1,\cdot\sigma_2)\|_{L^{p_1}(\sigma_1)\times L^{p_2}(\sigma_2)\to L^p(w)} \leq [w,\vec{\sigma}]_{A_{\vec{P}/p_0}}^{\frac{1}{p}}([\vec{\sigma}]_{H_{\vec{P}}^{\infty}}^{1/p} + \sum_{i=1}^{2}[\vec{\sigma}^i]_{H_{\vec{P}i}^{\infty}}^{1/p_i'}), \tag{11.10}$$

where $[\vec{\sigma}^i]_{H_{\vec{P}i}^{\infty}} = 1$ if $p \leq \gamma$ and otherwise,

$$[\vec{\sigma}^i]_{H_{\vec{P}i}^{\infty}} = \sup_Q \langle w\rangle_Q^{p_i'(\frac{1}{\gamma}-\frac{1}{p})_+} \exp\Big(\fint_Q w^{-1}\Big)^{p_i'(\frac{1}{\gamma}-\frac{1}{p})_+}$$
$$\times \prod_{j\neq i}\langle\sigma_i\rangle_Q^{p_i'/p_j} \exp\Big(\fint_Q \sigma_i^{-1}\Big)^{p_i'/p_j},$$

and

$$[\vec{w}]_{H_{\vec{P}}^{\infty}} := \sup_Q \prod_{i=1}^{m}\langle w_i\rangle_Q^{\frac{p}{p_i}} \exp\Big(\fint_Q \log w_i^{-1}\Big)^{\frac{p}{p_i}}.$$

If $\gamma < p_0$, then the above result still holds for all $p > \gamma$.

Proof. Likewise, we only study the estimate of I. Again,

$$\begin{aligned}
&\Big\| \sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\mathbf{1}_Q\Big\|_{L^q(w)}\\
&\leq [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \langle\sigma_1\rangle_Q^{\frac{p}{p_1}}\langle\sigma_2\rangle_Q^{\frac{p}{p_2}}|Q|\Big)^{\gamma/p}\\
&\leq [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p}}[\vec{\sigma}]_{H_{\vec{P}}^{\infty}}^{\frac{\gamma}{p}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \prod_{i=1}^{2}\exp\Big(\fint_Q \log\sigma_i\Big)^{\frac{p}{p_i}}|Q|\Big)^{\gamma/p}\\
&\leq [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p}}[\vec{\sigma}]_{H_{\vec{P}}^{\infty}}^{\frac{\gamma}{p}}\prod_{i=1}^{2}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \exp\Big(\fint_Q \log\sigma_i\Big)|Q|\Big)^{\gamma/p_i}\\
&\lesssim [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p}}[\vec{\sigma}]_{H_{\vec{P}}^{\infty}}^{\frac{\gamma}{p}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \exp\Big(\fint_Q \log\sigma_1\Big)|Q|\Big)^{\gamma/p_1}\|M_0(\mathbf{1}_{F_2}\sigma_2)\|_{L^1}^{\frac{\gamma}{p_2}}\\
&\leq [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{\gamma}{p}}[\vec{\sigma}]_{H_{\vec{P}}^{\infty}}^{\frac{\gamma}{p}}\sigma_2(F)^{\frac{\gamma}{p_2}}\Big(\sum_{\substack{Q\in\mathcal{S}\\ \pi_{\mathcal{F}_2}(Q)=F_2}} \exp\Big(\fint_Q \log\sigma_1\Big)|Q|\Big)^{\gamma/p_1},
\end{aligned}$$

where

$$M_0(f) := \sup_Q \exp\Big(\fint_Q \log|f|\Big)\mathbf{1}_Q, \tag{11.11}$$

is the logarithmic maximal function. Here we have used the fact that this maximal function is bounded from L^p into itself for $p \in (0,\infty)$ with bound independent of the dimension in the dyadic case as proved in [25, Lemma 2.1]. Hence,

$$
\begin{aligned}
I &\le [w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}[\vec\sigma]_{H^\infty_{\vec P}}^{\gamma/p}\Big(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q \sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^q\sigma_2(F)^{\frac{p}{p_2}}\\
&\quad\times\Big(\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\exp\Big(\fint_Q\log\sigma_1\Big)|Q|\Big)^{p/p_1}\Big)^{\frac{\gamma}{p}}\\
&\le [w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}[\vec\sigma]_{H^\infty_{\vec P}}^{\gamma/p}\Big(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^q\Big(\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^{p_2/\gamma}\sigma_2(F)\Big)^{\frac{p}{p_2}}\\
&\quad\times\Big(\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\exp\Big(\fint_Q\log\sigma_1\Big)|Q|\Big)^{p/p_1}\Big)^{\frac{\gamma}{p}}\\
&\le [w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}[\vec\sigma]_{H^\infty_{\vec P}}^{\gamma/p}\Big(\sum_{F_1\in\mathcal F_1}(\langle f_1\rangle_{F_1}^{\sigma_1})^{p_1/\gamma}\Big(\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}\sum_{\substack{Q\in\mathcal S\\ \pi_{\mathcal F_2}(Q)=F_2}}\exp\Big(\fint_Q\log\sigma_1\Big)|Q|\Big)\Big)^{\frac{\gamma}{p_1}}\\
&\quad\times\Big(\sum_{F_1\in\mathcal F_1}\sum_{\substack{F_2\in\mathcal F_2\\ \pi_{\mathcal F_1}(F_2)=F_1}}(\langle f_2\rangle_{F_2}^{\sigma_2})^{p_2/\gamma}\sigma_2(F)\Big)^{\frac{\gamma}{p_2}}\\
&\le [w,\vec\sigma]_{A_{\vec P}}^{\frac{\gamma}{p}}[\vec\sigma]_{H^\infty_{\vec P}}^{\gamma/p}\|f_1\|_{L^{p_1/\gamma}(\sigma_1)}\|f_2\|_{L^{p_2/\gamma}(\sigma_2)}.
\end{aligned}
$$

□

12. Applications

12.1. Mixed A_p-A_∞ estimate for commutators of multilinear Calderón-Zygmund operators. Throughout this section, we will work with commutators of multilinear Calderón-Zygmund operators with symbols in BMO. Recall that BMO consists of all locally integrable functions b with $||b||_{BMO} < \infty$, where

$$\|b\|_{BMO} := \sup_Q \frac{1}{|Q|}\int_Q |b(y) - \langle b\rangle_Q|dy,$$

and the supremum in the above definition is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes.

Given a multilinear Calderón-Zygmund operator T and $\vec b \in BMO^m$, we consider the following commutators with $\vec b$,

$$[\vec b, T] = \sum_{i=1}^m [\vec b, T]_i,$$

where

$$[\vec b, T]_i(\vec f) := b_i T(\vec f) - T(f_1, \cdots, f_{i-1}, b_i f_i, f_{i+1}, \cdots, f_m).$$

Our aim in this section is to prove the following mixed estimate for commutators of multilinear Calderón-Zygmund operators following the same spirit as in [10].

Theorem 12.1. *Let T be a multilinear Calderón-Zygmund operator and $\vec{b} \in BMO^m$. If we assume that $[w,\vec{\sigma}]_{A_{\vec{P}}} < \infty$, then*

$$\begin{aligned}&\|[\vec{b},T]\|_{L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)\to L^p(w)}\\&\quad\le [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{1}{p}}\Big(\prod_{i=1}^m[\sigma_i]_{A_\infty}^{\frac{1}{p_i}}+[w]_{A_\infty}^{\frac{1}{p'}}\sum_{j=1}^m\prod_{i\neq j}[\sigma_i]_{A_\infty}^{\frac{1}{p_i}}\Big)\\&\quad\times\Big([w]_{A_\infty}+\sum_{i=1}^m[\sigma_i]_{A_\infty}\Big)\left(\sum_{i=1}^m\|b_i\|_{BMO}\right),\end{aligned}$$

where $\sigma_i = w_i^{1-p_i'}$, $i = 1,\dots,m$.

Before proving our main result in this section we need to recall some basic properties about BMO functions and A_∞ weights that we are going to use in the sequel. Recall that a key property of BMO functions is the celebrated John-Nirenberg inequality [28].

Proposition 12.2. [29, pp. 31-32] *There are dimensional constants $0 < \alpha_n < 1 < \beta_n < \infty$ such that*

$$\sup_Q \frac{1}{|Q|}\int_Q \exp\Big(\frac{\alpha_n}{\|b\|_{\mathrm{BMO}}}|b(y)-\langle b\rangle_Q|\Big)dy \le \beta_n. \tag{12.1}$$

In fact, we can take $\alpha_n = \frac{1}{2^{n+2}}$.

It is well-known that if $w \in A_\infty$, then $\log w \in BMO$. Using the John-Nirenberg inequality, Chung, Pereyra, and Pérez [10] proved the following bound.

Proposition 12.3. *Let $b \in BMO$ and let $0 < \alpha_n < 1 < \beta_n < \infty$ be the dimensional constants from* (12.1)*. Then*

$$s\in\mathbb{R},\ |s| \le \frac{\alpha_n}{\|b\|_{BMO}}\min\{1,\frac{1}{p-1}\} \Rightarrow e^{sb}\in A_p \text{ and } [e^{sb}]_{A_p}\le \beta_n^p.$$

In [25], Hytönen and Pérez also showed the following bound for the Fujii-Wilson A_∞ constant of a particular family of weights.

Proposition 12.4. *There are dimensional constants ε_n and c_n such that*

$$[e^{\mathrm{Re} zb}w]_{A_\infty}\le c_n[w]_{A_\infty} \qquad \text{if } |z| \le \frac{\varepsilon_n}{\|b\|_{BMO}[w]_{A_\infty}}.$$

For our purpose, we need to show the following variation of the previous lemmas.

Lemma 12.5. *Suppose that $[w,\vec{\sigma}]_{A_{\vec{P}}} < \infty$ and $w, \sigma_i \in A_\infty$, $i = 1, 2\cdots, m$. Then for any $1 \le j \le m$,*

$$[we^{pb\mathrm{Re}z},\sigma_1,\cdots,\sigma_j e^{-p_j' b\mathrm{Re}z},\cdots,\sigma_m]_{A_{\vec{P}}}\le c_{n,\vec{P}}[w,\vec{\sigma}]_{A_{\vec{P}}},$$

provided that

$$|z|\le\frac{\alpha_n\min\{1,\frac{p_1'}{p},\cdots,\frac{p_m'}{p}\}}{p(1+\max\{[w]_{A_\infty},[\sigma_1]_{A_\infty},\cdots,[\sigma_m]_{A_\infty}\})\|b\|_{BMO}}.$$

To prove the previous lemma, we need to recall this sharp version of the reverse Hölder's inequality proved in [25].

Proposition 12.6. *Let $w \in A_\infty$. Then for any $0 \le r \le 1 + \frac{1}{c_n[w]_{A_\infty}}$, we have*

$$\Big(\frac{1}{|Q|}\int_Q w(x)^r dx\Big)^{\frac{1}{r}} \le 2\frac{1}{|Q|}\int_Q w(x)dx.$$

Proof of Lemma 12.5. Set

$$r = 1 + \frac{1}{c_n \max\{[w]_{A_\infty}, [\sigma_j]_{A_\infty}\}}.$$

By definition of the $A_{\vec{P}}$ constant, Hölder's inequality and Proposition 12.6, we have

$$\begin{aligned}
&[we^{pb\mathrm{Re}z}, \sigma_1, \cdots, \sigma_j e^{-p_j' b\mathrm{Re}z}, \cdots, \sigma_m]_{A_{\vec{P}}} \\
&= \sup_Q \langle we^{pb\mathrm{Re}z}\rangle_Q \langle \sigma_j e^{-p_j' b\mathrm{Re}z}\rangle_Q^{\frac{p}{p_j'}} \prod_{i\ne j}\langle\sigma_i\rangle_Q^{\frac{p}{p_i'}} \\
&\le \sup_Q \langle w^r\rangle_Q^{\frac{1}{r}} \langle e^{pbr'\mathrm{Re}z}\rangle_Q^{\frac{1}{r'}} \langle \sigma_j^r\rangle_Q^{\frac{p}{rp_j'}} \langle e^{-p_j' br'\mathrm{Re}z}\rangle_Q^{\frac{p}{r'p_j'}} \prod_{i\ne j}\langle\sigma_i\rangle_Q^{\frac{p}{p_i'}} \\
&\le 4\sup_Q \langle w\rangle_Q \langle e^{pbr'\mathrm{Re}z}\rangle_Q^{\frac{1}{r'}} \langle \sigma_j\rangle_Q^{\frac{p}{p_j'}} \langle e^{-p_j' br'\mathrm{Re}z}\rangle_Q^{\frac{p}{r'p_j'}} \prod_{i\ne j}\langle\sigma_i\rangle_Q^{\frac{p}{p_i'}} \\
&\le 4[w,\vec{\sigma}]_{A_{\vec{P}}} [e^{pbr'\mathrm{Re}z}]_{A_{1+\frac{p}{p_j'}}}^{\frac{1}{r'}} \\
&\le c_{n,\vec{P}}[w,\vec{\sigma}]_{A_{\vec{P}}},
\end{aligned}$$

where Proposition 12.3 is used in the last step. □

Now we are ready to prove the main result in this section.

Proof of Theorem 12.1. It suffices to study the boundedness of $[\vec{b}, T]_i$. Without loss of generality, we just consider the case $i = 1$. Using the same trick as that in [10, Thm. 3.1], for any complex number z, we define

$$T_z^1(\vec{f}) = e^{zb}T(e^{-zb}f_1, f_2, \cdots, f_m).$$

Then by using the Cauchy integral theorem, we get for "nice" functions,

$$[b,T]_1(\vec{f}) = \frac{d}{dz}T_z^1(\vec{f})\Big|_{z=0} = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{T_z^1(\vec{f})}{z^2}dz,\ \varepsilon > 0.$$

Next, using Minkowski's inequality, for $p \ge 1$,

$$\|[b,T]_1(\vec{f})\|_{L^p(w)} \le \frac{1}{2\pi\varepsilon^2}\int_{|z|=\varepsilon}\|T_z^1(\vec{f})\|_{L^p(w)}|dz|. \tag{12.2}$$

Notice that

$$\|T_z^1(\vec{f})\|_{L^p(w)} = \|T(e^{-zb}f_1, f_2, \cdots, f_m)\|_{L^p(we^{pb\mathrm{Re}z})}. \tag{12.3}$$

Therefore, applying the boundedness properties for Calderón–Zygmund operators in Theorem 11.1 for weights $(we^{pb\mathrm{Re}z}, w_1e^{p_1b\mathrm{Re}z}, w_2, \dots, w_m)$ with $p_0 = \gamma = 1$, we get

$$\begin{aligned}\|T(e^{-zb}f_1, f_2, \cdots, f_m)\|_{L^p(we^{pb\mathrm{Re}z})} &\lesssim [e^{pb\mathrm{Re}z}w, e^{-p_1'b\mathrm{Re}z}\sigma_1, \sigma_2, \dots, \sigma_m]^{1/p}_{A_{\vec{P}}} \\ &\times \Big([e^{-p_1'b\mathrm{Re}z}\sigma_1]^{1/p_1}_{A_\infty}\prod_{i=2}^m[\sigma_i]^{1/p_i}_{A_\infty} + [e^{pb\mathrm{Re}z}w]^{1/p'}_{A_\infty}\Big(\prod_{i=2}^m[\sigma_i]^{1/p_i}_{A_\infty} + \\ &+\sum_{i'=2}^m[\sigma_1e^{-p_1'b\mathrm{Re}z}]^{1/p_1}_{A_\infty}\prod_{\substack{i\neq i'\\ i>1}}[\sigma_i]^{1/p_i}_{A_\infty}\Big)\Big)||f_1e^{-zb}||_{L^{p_1}(e^{bp_1\mathrm{Re}z}w_1)}\prod_{i=2}^m||f_i||_{L^{p_i}(w_i)}.\end{aligned} \tag{12.4}$$

Combining (12.2), (12.3) and (12.4) and using Proposition 12.4 and Lemma 12.5, we arrive at

$$\begin{aligned}&\|[b,T]_1(\vec{f})\|_{L^p(w)} \\ &\leq \frac{1}{2\pi\varepsilon}[w,\vec{\sigma}]^{1/p}_{A_{\vec{P}}}\left(\prod_{i=1}^m[\sigma_i]^{1/p_i}_{A_\infty} + [w]^{1/p'}_{A_\infty}\sum_{i'=1}^m\prod_{i'\neq i}[\sigma_i]^{1/p_i}_{A_\infty}\right)\prod_{i=1}^m||f_i||_{L^{p_i}(w_i)}.\end{aligned} \tag{12.5}$$

Now taking

$$\varepsilon = \frac{c_{n,\vec{P}}}{([w]_{A_\infty} + \sum_{i=1}^m[\sigma_i]_{A_\infty})\|b_1\|_{BMO}},$$

where $c_{n,\vec{P}}$ is sufficiently small such that it satisfies the hypotheses in Proposition 12.4 and Lemma 12.5. Then,we obtain

$$\begin{aligned}\|[b,T]_1(\vec{f})\|_{L^p(w)} &\lesssim [w,\vec{\sigma}]^{\frac{1}{p}}_{A_{\vec{P}}}(\prod_{i=1}^m[\sigma_i]^{\frac{1}{p_i}}_{A_\infty} + [w]^{\frac{1}{p'}}_{A_\infty}\sum_{j=1}^m\prod_{i\neq j}[\sigma_i]^{\frac{1}{p_i}}_{A_\infty}) \\ &\times([w]_{A_\infty} + \sum_{i=1}^m[\sigma_i]_{A_\infty})||b_1||_{BMO}\prod_{i=1}^m||f_i||_{L^{p_i}(w_i)}.\end{aligned}$$

The general result follows immediately combining the estimates for all the commutators in the different variables. □

12.2. **Mixed A_p-A_∞ estimates for multilinear square functions and multilinear Fourier multipliers.** The results obtained in Section 11 can be applied to different instances of operators which can be reduced to the simpler dyadic operators $\mathcal{A}_{p_0,\gamma,\mathcal{S}}$.
Firstly, observe that the mixed weighted bounds obtained in the main theorems in Section 11 can be extended to the case of multilinear square functions taking into account [7, Prop. 4.2] and choosing $p_0 = 1$ and $\gamma = 2$.
These mixed bounds can also be extended to multilinear Fourier multipliers, which are a particular example of a general class of operators whose kernels satisfy weaker regularity conditions than the usual Hölder continuity. To obtain the corresponding mixed bounds, it is sufficient to consider the results in [6] together with the main theorems in Section 11 for $\gamma = 1$. It is worth mentioning that these mixed bounds for Fourier multipliers seem to be new in the multilinear scenario.

Appendix A. Unweigthed bounds

In this appendix we state and prove some well-known boundedness results for bilinear Calderón–Zygmund operators and their maximal truncations defined in Section 10, which also hold in the multilinear setting. It is worth mentioning that the novelty of these results is not only that they are stated in a quantitative way that will be useful for our purposes, but also that some of these results are proved under weaker regularity conditions on the kernels than those results in the literature.

Lemma A.1. *Let T be a bilinear Dini-continuous Calderón-Zygmund operator. Then T is bounded from $L^1 \times L^1$ to $L^{\frac{1}{2},\infty}$ and*

$$\|T\|_{L^1\times L^1\to L^{1/2,\infty}} \lesssim \|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\mathrm{Dini}}, \tag{A.1}$$

where $\|T\|_{L^{q_1}\times L^{q_2}\to L^q}$ denotes the norm of the operator as in its definition.

This result was proved under the $Dini(\frac{1}{2})$ condition in [42]. Observe that $Dini(1/2)$ condition is an stronger condition than Dini condition, which is also referred to as $Dini(1)$. In [46], Pérez and Torres studied the problem under the $BGHC$ condition. Namely, we say that a bilinear operator with kernel K satisfies the bilinear geometric Hörmander condition ($BGHC$) if there exists a fixed constant C such that and for any family of disjoint dyadic cubes D_1 and D_2,

$$\int_{\mathbb{R}^n} \sup_{y\in Q} \int_{\mathbb{R}\setminus Q^*} |K(x,y,z) - K(x,y_Q,z)|dxdz \leq C, \tag{A.2}$$

$$\int_{\mathbb{R}^n} \sup_{z\in P} \int_{\mathbb{R}\setminus P^*} |K(x,y,z) - K(x,y,z_P)|dxdy \leq C, \tag{A.3}$$

and

$$\sum_{(P,Q)\in D_1\times D_2} |P||Q| \sup_{(y,z)\in P\times Q} \int_{\mathbb{R}^n\setminus(\cup_{R\in D_1})\cup(\cup_{S\in D_2})} |K(x,y,z) - K(x,y_P,z_Q)|dx \leq C(|\cup_{P\in D_1} P| + |\cup_{Q\in D_2} Q|). \tag{A.4}$$

Here Q^* is the cube with the same center as Q and sidelength $10\sqrt{n}\ell(Q)$.This condition, which is actually stated here in an equivalent way, was shown to be weaker than the Dini condition in [46, Prop. 2.3]). Thus, Lemma A.1 follows immediately from the mentioned result. Here we give the proof with the precise constants.

Proof of Lemma A.1. Suppose that T is initially bounded from $L^{q_1} \times L^{q_2}$ to L^q, where $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. We shall dominate the bound $\|T\|_{L^1\times L^1\to L^{1/2,\infty}}$ by $\|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\mathrm{Dini}}$. Indeed, fix $\lambda > 0$ and consider without loss of generality functions $f_i \geq 0$, $i = 1, 2$. Let $\alpha_i > 0$ be numbers to be determined later. Apply the Calderón-Zygmund decomposition to f_i at height $\alpha_i\lambda$, to obtain its good and bad parts g_i and b_i, respectively, and families of cubes $\{Q_k^i\}_k$ with disjoint interiors such that $f_i = g_i + b_i$ and $b_i = \sum_k b_k^i$ verifying the properties in [20, Thm. 4.3.1].
Next, set $\Omega_i = \cup_k 4nQ_k^i$. We have

$$\left|\{x : |T(f_1,f_2)(x)| > \lambda\}\right| \leq |\Omega_1| + |\Omega_2| + \left|\{x \in (\Omega_1\cup\Omega_2)^c : |T(g_1,g_2)(x)| > \frac{\lambda}{4}\}\right|$$

$$+ \big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(g_1, b_2)(x)| > \frac{\lambda}{4}\}\big|$$
$$+ \big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(b_1, g_2)(x)| > \frac{\lambda}{4}\}\big|$$
$$+ \big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(b_1, b_2)(x)| > \frac{\lambda}{4}\}\big|.$$

It is easy to see that

$$|\Omega_1| + |\Omega_2| \leq C_n\Big(\frac{1}{\alpha_1\lambda}\|f_1\|_{L^1} + \frac{1}{\alpha_2\lambda}\|f_2\|_{L^1}\Big).$$

For the third term, using Chebychev's inequality and the boundedness properties of T and g_i, we have

$$\begin{aligned}
&\big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(g_1, g_2)(x)| > \frac{\lambda}{4}\}\big| \\
&\leq \frac{4^q}{\lambda^q}\|T(g_1, g_2)\|_{L^q}^q \\
&\leq \frac{4^q}{\lambda^q}\|T\|_{L^{q_1}\times L^{q_2}\to L^q}^q \|g_1\|_{L^{q_1}}^q \|g_2\|_{L^{q_2}}^q \\
&\leq \frac{4^q}{\lambda^q} C_{n,q,q_1,q_2}\|T\|_{L^{q_1}\times L^{q_2}\to L^q}^q (\alpha_1\lambda)^{q/q_1'}(\alpha_2\lambda)^{q/q_2'}\|f_1\|_{L^1}^{q/q_1}\|f_2\|_{L^1}^{q/q_2}.
\end{aligned}$$

For the fourth term, if c_k denotes the center of the cube Q_k^2, we have

$$\begin{aligned}
&\big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(g_1, b_2)(x)| > \frac{\lambda}{4}\}\big| \\
&\leq \frac{4}{\lambda}\int\Big|\sum_k \int\int_{Q_k^2}(K(x,y,z) - K(x,y,c_k))g_1(y)b_2^k(z)dzdy\Big|dx \\
&\leq \frac{4}{\lambda}\sum_k \int\int\int_{Q_k^2}|K(x,y,z) - K(x,y,c_k)|\cdot|g_1(y)|\cdot|b_2^k(z)|dzdydx \\
&\leq \frac{4}{\lambda}\sum_k \int_{Q_k^2}\int\int \omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2(|x-y|+|x-z|)}\Big)\frac{|g_1(y)|\cdot|b_2^k(z)|}{(|x-y|+|x-z|)^{2n}}dydxdz \\
&\leq C_n\alpha_1\sum_k \int_{Q_k^2}\int\int \omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2(|y|+|x-z|)}\Big)\frac{|b_2^k(z)|}{(|y|+|x-z|)^{2n}}dydxdz \\
&\leq C_n\alpha_1\sum_k \int_{Q_k^2}\int\int \omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2|x-z|}\Big)\frac{|b_2^k(z)|}{(|y|+|x-z|)^{2n}}dydxdz \\
&\leq C_n\alpha_1\sum_k \int_{Q_k^2}\int_{|x-z|\geq n\ell(Q_k^2)} \omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2|x-z|}\Big)\frac{|b_2^k(z)|}{|x-z|^n}dxdz \\
&\leq C_n'\alpha_1\|\omega\|_{\text{Dini}}\|f_2\|_{L^1},
\end{aligned}$$

where we have used the cancelation properties of b_2^k, the regularity condition on the third variable of K (since $|z-c_k| < \tau\max(|x-y|, |x-z|)$ for $x \notin \Omega_1 \cup \Omega_2$), the fact that ω is increasing, the Dini condition, $||g_1||_{L^\infty} \leq c_n\alpha_1\lambda$ and $\sum_k ||b_2^k||_{L^1} \leq c_n||f_2||_{L^1}$.

Since the estimate of the fifth term is symmetric to the previous estimate, it remains to estimate the last term. If we denote as c_l and c_k the center of the cubes Q_l^1 and Q_k^2, respectively, proceeding similarly as in the previous estimate, we obtain

$$
\begin{aligned}
&\Big|\{x\in(\Omega_1\cup\Omega_2)^c:|T(b_1,b_2)(x)|>\frac{\lambda}{4}\}\Big|\\
&\le\frac{4}{\lambda}\int\Big|\sum_{k,l}\int_{Q_l^1}\int_{Q_k^2}(K(x,y,z)-K(x,y,c_k))b_1^l(y)b_2^k(z)dzdy\Big|dx\\
&\le\frac{4}{\lambda}\sum_{k,l}\int_{(\Omega_1\cup\Omega_2)^c}\int_{Q_l^1}\int_{Q_k^2}|K(x,y,z)-K(x,y,c_k)||b_1^l(y)||b_2^k(z)|dxdydx\\
&\le\frac{4}{\lambda}\sum_{k,l}\int_{Q_k^2}\int_{Q_l^1}\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2(|x-y|+|x-z|)}\Big)\frac{|b_1^l(y)||b_2^k(z)|dxdydz}{(|x-y|+|x-z|)^{2n}}\\
&\le\frac{C_n}{\lambda}\sum_{k,l}\int_{Q_k^2}\int_{Q_l^1}\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2(|x-c_l|+|x-c_k|)}\Big)\frac{|b_1^l(y)||b_2^k(z)|dxdydz}{(|x-c_l|+|x-c_k|)^{2n}}\\
&\le C_n\sum_{k,l}|Q_l^1||Q_k^2|\alpha_1\alpha_2\lambda\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}(\ell(Q_k^2)+\ell(Q_l^1))}{2(|x-c_l|+|x-c_k|)}\Big)\frac{dx}{(|x-c_l|+|x-c_k|)^{2n}}\\
&\le C_n'\sum_{k,l}\alpha_1\alpha_2\lambda\int_{Q_k^2}\int_{Q_l^1}\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}(\ell(Q_k^2)+\ell(Q_l^1))}{2(|x-y|+|x-z|)}\Big)\frac{dxdydz}{(|x-y|+|x-z|)^{2n}}\\
&=C_n'\sum_{k,l}\alpha_1\alpha_2\lambda\Big(\int_{\ell(Q_k^2)\ge\ell(Q_l^1)}+\int_{\ell(Q_l^1)\ge\ell(Q_k^2)}\Big)\\
&\le I+II.
\end{aligned}
$$

By symmetry, it suffices to estimate I. We have

$$
\begin{aligned}
I&\le C_n'\sum_k\alpha_1\alpha_2\lambda\int_{Q_k^2}\int_{(\Omega_1\cup\Omega_2)^c}\int_{\mathbb{R}^n}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2))}{|x-z|}\Big)\frac{dydxdz}{(|x-y|+|x-z|)^{2n}}\\
&=C_n'\sum_k\alpha_1\alpha_2\lambda\int_{Q_k^2}\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2))}{|x-z|}\Big)\frac{1}{|x-z|^n}dxdz\\
&\le C_n\alpha_1\|\omega\|_{\mathrm{Dini}}\|f_2\|_{L^1}.
\end{aligned}
$$

Combining the arguments above, we have

$$
\begin{aligned}
&\Big|\{x:|T(f_1,f_2)(x)|>\lambda\}\Big|\\
&\qquad\lesssim\frac{1}{\alpha_1\lambda}\|f_1\|_{L^1}+\frac{1}{\alpha_2\lambda}\|f_2\|_{L^1}\\
&\qquad+\|T\|^q_{L^{q_1}\times L^{q_2}\to L^q}(\alpha_1)^{q/q_1'}(\alpha_2)^{q/q_2'}\lambda^{q-1}\|f_1\|_{L^1}^{q/q_1}\|f_2\|_{L^1}^{q/q_2}\\
&\qquad+\alpha_1\|\omega\|_{\mathrm{Dini}}\|f_2\|_{L^1}+\alpha_2\|\omega\|_{\mathrm{Dini}}\|f_1\|_{L^1}
\end{aligned}
$$

Take

$$
\alpha_1=\lambda^{-\frac12}\frac{\|f_1\|_{L^1}^{\frac12}}{\|f_2\|_{L^1}^{\frac12}}\frac{1}{(\|T\|_{L^{q_1}\times L^{q_2}\to L^q}+\|\omega\|_{\mathrm{Dini}})^{\frac12}}
$$

$$\alpha_2 = \lambda^{-\frac{1}{2}} \frac{\|f_2\|_{L^1}^{\frac{1}{2}}}{\|f_1\|_{L^1}^{\frac{1}{2}}} \frac{1}{(\|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\text{Dini}})^{\frac{1}{2}}},$$

we get

$$\lambda\big|\{x : |T(f_1, f_2)(x)| > \lambda\}\big|^2 \le (\|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\text{Dini}})\|f_1\|_{L^1}\|f_2\|_{L^1}.$$

□

We also need to show that the maximal truncated operator $T_\sharp$ is bounded from $L^1 \times L^1$ to $L^{\frac{1}{2},\infty}$. Therefore, we need to check first that Cotlar's inequality holds for this class of operators.

Theorem A.2. *Let T be a bilinear Dini-continuous Calderón-Zygmund operator with kernel K. Then, for all $\eta \in (0, \frac{1}{2})$, there exists a constant C such that*

$$T_\sharp(\vec{f}) \le c_{\eta,n}(C_K + ||\omega||_{\text{Dini}} + ||T||_{L^{q_1}\times L^{q_2}\to L^q})\mathcal{M}(\vec{f}) + M_\eta(|T(\vec{f})|). \tag{A.5}$$

In this proof we combine the strategies used in [42, Thm 6.4] and [26, Lemma 5.3] to determine the precise constants involved in the inequality.

Proof of Theorem A.2. Let us begin defining the following maximal truncation

$$\widetilde{T}_\sharp(f_1, f_2)(x) = \sup_{\varepsilon>0} \Big|\widetilde{T}_\varepsilon(f_1, f_2)(x)\Big|,$$

where

$$\widetilde{T}_\varepsilon(f_1, f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} K(x, y, z)f_1(y)f_2(z)dydz.$$

Since

$$\sup_{\varepsilon>0} \left| \int_{\substack{\max\{|x-y|,|x-z|\}\le\varepsilon \\ |x-y|^2+|x-z|^2>\varepsilon^2}} K(x, y, z)f_1(y)f_2(z)dydz \right| \lesssim C_K\mathcal{M}(f_1, f_2)(x), \tag{A.6}$$

it suffices to show (A.5) with $T_\sharp$ replaced by $\widetilde{T}_\sharp$. Notice that we can write for $x' \in B(x, \varepsilon/2)$,

$$\begin{aligned} \widetilde{T}_\varepsilon(f_1, f_2)(x) = &\int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x, y, z) - K(x', y, z))f_1(y)f_2(z)dydz \\ &+ T(f_1, f_2)(x') - T(f_1^0, f_2^0)(x'), \end{aligned} \tag{A.7}$$

where $f_i^0 = f_i\mathbf{1}_{B(x,\varepsilon)}$. For the first term in (A.7), using the regularity assumptions on the kernel, we get

$$\begin{aligned} &\Big|\int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x, y, z) - K(x', y, z))f_1(y)f_2(z)dydz\Big| \\ &\le \int_{\max\{|x-y|,|x-z|\}>\varepsilon} \omega\Big(\frac{|x - x'|}{|x - y| + |x - z|}\Big) \frac{|f_1(y)||f_2(z)|dydz}{(|x - y| + |x - z|)^{2n}} \\ &= \sum_{k=0}^{\infty} \int_{2^k\varepsilon<\max\{|x-y|,|x-z|\}\le 2^{k+1}\varepsilon} \omega\Big(\frac{|x - x'|}{2^k\varepsilon}\Big) \frac{1}{(2^k\varepsilon)^{2n}} |f_1(y)||f_2(z)|dydz \end{aligned}$$

$$
\begin{aligned}
&\lesssim \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\omega\Big(\frac{|x-x'|}{2^k\varepsilon}\Big)\\
&\lesssim \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\int_{2^{k-1}}^{2^k}\omega(\frac{|x-x'|}{\varepsilon t})\frac{dt}{t}\\
&= \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\int_{\frac{|x-x'|}{2^k\varepsilon}}^{\frac{|x-x'|}{2^{k-1}\varepsilon}}\omega(u)\frac{du}{u}\\
&= \mathcal{M}(f_1,f_2)(x)\int_0^{\frac{2|x-x'|}{\varepsilon}}\omega(u)\frac{du}{u}\\
&\le \|\omega\|_{\mathrm{Dini}}\mathcal{M}(f_1,f_2)(x),
\end{aligned}
$$

where the last step holds since $|x-x'|\le\varepsilon/2$. Next, taking the L^η average over $x'\in B(x,\varepsilon/2)$, we arrive at

$$
\begin{aligned}
|\widetilde{T}_\varepsilon(f_1,f_2)(x)| \lesssim{}& ||\omega||_{\mathrm{Dini}}\mathcal{M}(f_1,f_2)(x)+M_\eta(|T(f_1,f_2)|)(x)\\
&+\left(\frac{1}{|B(x,\varepsilon/2)|}\int_{B(x,\varepsilon/2)}|T(f_1^0,f_2^0)(x')|^\eta dx'\right)^{1/\eta}.
\end{aligned}
$$

For the last term, using Kolmogorov's inequality to relate the L^η and $L^{1/2,\infty}$ norms and the boundedness of T from $L^1\times L^1$ to $L^{1/2,\infty}$, we obtain for any $\eta\in(0,\frac{1}{2})$,

$$
\begin{aligned}
\Big(\frac{1}{|B(x,\varepsilon/2)|}&\int_{B(x,\varepsilon/2)}|T(f_1^0,f_2^0)(x')|^\eta dx'\Big)^{1/\eta}\\
&= ||T(f_1^0,f_2^0)||_{L^\eta(B(x,\frac{\varepsilon}{2}),\frac{dx}{|B(x,\frac{\varepsilon}{2})|})}\\
&\le C_\eta||T(f_1^0,f_2^0)||_{L^{1/2,\infty}(B(x,\frac{\varepsilon}{2}),\frac{dx}{|B(x,\frac{\varepsilon}{2})|})}\\
&\le C_\eta||T||_{L^1\times L^1\to L^{1/2,\infty}}\mathcal{M}(f_1,f_2)(x).
\end{aligned}
$$

Combining all the terms, we finally arrive at

$$
\begin{aligned}
|\widetilde{T}_\varepsilon(f_1,f_2)(x)| \le{}& c_n(||\omega||_{\mathrm{Dini}}+C_\eta||T||_{L^1\times L^1\to L^{1/2,\infty}})\mathcal{M}(f_1,f_2)(x)\\
&+M_\eta(|T(f_1,f_2)|)(x),
\end{aligned}
$$

which taking into account (A.6) and (A.1) leads to the desired result. □

As a corollary of the previous result follows the weak boundedness of the maximal truncation of T.

Corollary A.3. *Let T be a bilinear Calderón–Zygmund operator with Dini-continuous kernel K. Then*

$$
||T_\sharp||_{L^1\times L^1\to L^{1/2,\infty}} \lesssim (C_K+||\omega||_{\mathrm{Dini}}+||T||_{L^{q_1}\times L^{q_2}\to L^q}). \tag{A.8}
$$

Proof. Fix $\eta\in(0,1/2)$ and use the previous result together with the weak boundedness of the multilinear maximal function and bilinear Calderón–Zygmund operators and the fact that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$. To prove the latter, notice that for the Hardy-Littlewood maximal function using [26, Lemma 2.2], we can write

$$M(f) \eqsim \sum_{u=1}^{3^n} M_u(f),$$

where

$$M_u(f) := \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}^u}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Therefore,

$$\Big|\{x : M(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda\}\Big| \le \sum_{u=1}^{3^n} \Big|\{x : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}\Big|.$$

Denote

$$E_u := \{x \in \mathbb{R}^n : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}.$$

We can find a collection of maximal dyadic cubes $\{Q_j\}_j$ such that $E_u = \cup_j Q_j$ and

$$\frac{1}{|Q_j|} \int_{Q_j} |T(f_1,f_2)|^\eta > \lambda^\eta (3^n)^{-\eta},$$

which means that

$$|E_u| \le (3^n)^\eta \lambda^{-\eta} \int_{E_u} |T(f_1,f_2)|^\eta, \quad u = 1,\dots,3^n.$$

Now using Kolmogorov's inequality and the fact that $T : L^1 \times L^1 \to L^{1/2,\infty}$, and assuming that $\eta < 1/2$, we get

$$\int_{E_u} |T(f_1,f_2)|^\eta \lesssim \|T(f_1,f_2)\|^\eta_{L^{\frac{1}{2},\infty}(E_u,\frac{dx}{|E_u|})} |E_u| \le \|f_1\|_1^\eta \|f_2\|_1^\eta |E_u|^{1-2\eta}$$

Combining both estimates, it follows that

$$|E_u| \le \lambda^{-\eta}(3^n)^\eta \|f_1\|_1^\eta \|f_2\|_1^\eta |E_u|^{1-2\eta},$$

which is exactly,

$$\lambda |E_u|^2 \le c_{n,\eta} \|f_1\|_1 \|f_2\|_1.$$

□

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