Spring 2016

Weighted inequalities for multilinear singular integrals

Muckenhoupt's A_p weights

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Weights

DEFINITION

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere.

NOTATION

- Given a weight w and a measurable set E, we use the notation

$$w(E) = \int_E w(x)dx,$$

to denote the w-measure of the set E.

- The weighted Lebesgue spaces are denoted by $L^p(\mathbb{R}^n, w)$ or simply $L^p(w)$.

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Muckenhoupt A_p classes of weights

ONE-WEIGHT PROBLEM

Characterization of all weights $w(x)$ such that the strong type (p,p) inequality

$$\int_{\mathbb{R}^n} M(f)(x)^p w(x)dx \leq C_p^p \int_{\mathbb{R}^n} |f(x)|^p w(x)dx,$$

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A_p WEIGHTS

Let $1 < p < \infty$. A weight w is said to be of class A_p if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty.$$

The quantity $[w]_{A_p}$ is known as the A_p characteristic constant of the weight w.

A_1 class of weights

In the case $p = 1$, we also have the corresponding result

$$M : L^1(w) \to L^{1,\infty}(w),$$

if and only if w satisfies the A_1 condition, i.e.,

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A_1 CONDITION

If w is an A_1 weight, then the (finite) quantity

$$[w]_{A_1} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(t)dt \right) ||w^{-1}||_{L^\infty(Q)}$$

is called the A_1 characteristic constant of w.

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REMARK

If $w \in A_1$, then

$$\left(\frac{1}{|Q|}\int_Q w(t)dt\right) \leq [w]_{A_1} \operatorname{ess.inf}_{y\in Q} w(y),$$

for all cubes Q in $\mathbb{R}^n$.

Properties of A_p classes

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- **Reverse Hölder's inequality** There exist constants C and γ such that for every cube Q,

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- **Factorization** $w \in A_p$, $1 < p < \infty \Leftrightarrow \exists w_1, w_2 \in A_1$ such that $w = w_1 w_2^{1-p}$.
- **Extrapolation** An estimate on $L^{p_0}(v)$ for a single p_0 and all A_{p_0} weights v implies a similar $L^p(w)$ estimate for all $p \in (1,\infty)$ and all weights $w \in A_p$.

A_∞ class of weights

Observe that since the A_p classes are increasing, it is natural to define the A_∞ class of weights as

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HRUSČĚV A_∞ CONSTANT

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FUJII-WILSON A_∞ CONSTANT

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x)dx$$

Sharp Reverse Hölder Inequality for A_∞ weights

THEOREM [HP]

If $w \in A_\infty$, then

$$\left(\frac{1}{|Q|}\int_Q w^{r(w)}\right)^{1/r(w)} \leq 2\frac{1}{|Q|}\int_Q w,$$

where $r(w) = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$ and $\tau_n = 2^{11+n}$. Furthermore, $[w]_{A_\infty} \simeq r'(w)$.

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For interested readers

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