SPRING 2016

WEIGHTED INEQUALITIES FOR MULTILINEAR SINGULAR INTEGRALS

INTRODUCTION AND MOTIVATION

WENDOLÍN DAMIÁN GONZÁLEZ

14 MARCH 2016

Course information

Information

- **Mondays** and **Tuesdays**, 14-16, in room C122.
- Last day: 3 May, 2016.
- **Easter break**: No classes on 28-29 March, 2016.

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Give a short but detailed introduction to multilinear weighted inequalities and the usual techniques of proof in the area.

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Give a short but detailed introduction to multilinear weighted inequalities and the usual techniques of proof in the area.

PREREQUISITES

- Real and functional analysis, measure and integration.
- Basic inequalities such us Hölder and Minkowski.
- Linear weighted theory (desirable, but not compulsory).

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Instructors

- Wendolín Damián (B410): 14.03 - 11.04
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Exercises

- Set 1 > Deadline: 11.04
- Set 2 > Deadline: TBA

Recommended bibliography

- Grafakos, Loukas. Classical Fourier analysis. Graduate Texts in Mathematics, 249, (2014).
- Duoandikoetxea, Javier. Fourier analysis. Graduate Studies in Mathematics, 29, (2001).
- García-Cuerva, José, and Rubio de Francia, José L. Weighted norm inequalities and related topics. North-Holland Mathematics Studies, 116. Notas de Matemtica 104, (1985).

Framework: Lebesgue spaces

Lebesgue spaces

$L^p(\mathbb{R}^n,\mu)$, $1 \leq p < \infty$, is defined as the set of all μ-measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$ whose p-th powers are integrable, equipped with the norm

$$||f||_{L^p(\mathbb{R}^n,\mu)} = \left(\int_{\mathbb{R}^n} |f|^p d\mu \right)^{\frac{1}{p}}.$$

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Hölder's inequality

Let $p_1, \ldots, p_m, p$ be numbers such that

$$\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}.$$

Then

$$||f_1 \cdot \ldots \cdot f_m||_{L^p(\mathbb{R}^n,\mu)} \leq \prod_{i=1}^{m} ||f_i||_{L^{p_i}(\mathbb{R}^n,\mu)}.$$

Framework: Weak Lebesgue spaces

Weak Lebesgue spaces

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$$||f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} = \sup\{t > 0 : t\mu(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p}\} < \infty.$$

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Hölder's inequality for weak spaces

Let $f_j \in L^{p_j,\infty}(\mathbb{R}^n,\mu)$ where $0 < p_j < \infty$ for $j = 1,\ldots,k$. Let

$$\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}.$$

Then

$$||f_1 \ldots f_j||_{L^{p,\infty}(\mathbb{R}^n,\mu)} \leq p^{-1/p} \prod_{j=1}^{k} p_j^{1/p_j} ||f_j||_{L^{p_j}(\mathbb{R}^n,\mu)}.$$

Useful inequalities

Kolmogorov's inequality

Let $0 < p < q < \infty$. Then, there exists a constant $C = C_{p,q}$ such that for any measurable function f,

$$||f||_{L^p(Q,\frac{dx}{|Q|})} \leq C||f||_{L^{q,\infty}(Q,\frac{dx}{|Q|})},$$

where $C = \mathcal{O}\left(\frac{1}{q-p}\right)$.

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MINKOWSKI'S INEQUALITY

Let f be an integrable function on the product space $(\mathbb{R}^n, \mu) \times (\mathbb{R}^n, \nu)$ where μ, ν are σ-finite and $p \geq 1$. Then,

$$\left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} |f(x,y)| d\mu(x)\right|^p d\nu(y)\right]^{1/p} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x,y)|^p d\nu(y)\right]^{1/p} d\mu(x).$$

Weak and strong norm inequalities

DEFINITION

Let (X,μ) and (Y,ν) be measure spaces and let T be an operator defined from $L^p(X,\mu)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that:

1. T is *strong* (p,q) if $||Tf||_{L^q(Y,\nu)} \lesssim ||f||_{L^p(X,\mu)}$.
2. T is *weak* (p,q) if $||Tf||_{L^{q,\infty}(Y,\nu)} \lesssim ||f||_{L^p(X,\mu)}$.

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2. T is *weak* (p,q) if $||Tf||_{L^{q,\infty}(Y,\nu)} \lesssim ||f||_{L^p(X,\mu)}$.

When $(X,\mu)=(Y,\nu)$ in the above definition of weak (p,p) operator, we get the **Chebyshev's inequality**,

$$\mu(\{x\in X : |Tf(x)|>\lambda\}) \lesssim \left(\frac{||f||_{L^p(X,\mu)}}{\lambda}\right)^p.$$

Norm weighted inequalities

Goal

Determine under which conditions a given operator T (initially bounded on $L^p(\mathbb{R}^n, dx)$) satisfies is bounded on $L^p(\mathbb{R}^n, \mu)$, where $d\mu = w(x)dx$.

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We will say that w is a **weight** if it is a measurable locally integrable function defined in $\mathbb{R}^n$ taking values in $(0, \infty)$ for almost each point.

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Main operators under study

- Maximal functions (Hardy-Littlewood, fractional versions,...).
- Singular integral operators (CZO).

Origin of modern theory of weights

HARDY–LITTLEWOOD MAXIMAL FUNCTION

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THEOREM [MU]

For $1 < p < \infty$ it holds that

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx, \quad f \in L^p(w),$$

if and only if w satisfies the A_p condition, i.e.,

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

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THEOREM [MU]

For $1 \leq p < \infty$, it holds that

$$\sup_{\lambda > 0} \lambda^p \int_{\{Mf > \lambda\}} u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad f \in L^p(v).$$

if and only if

$$[u,v]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q u(x)dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

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THEOREM [SA]

Let (u,v) be weights. Then, for $1<p<\infty$ it holds that

$$M : L^p(v) \longrightarrow L^p(u),$$

if and only if

$$[u,v]_{S_p} = \sup_Q \left(\frac{\int_Q M(\chi_Q \sigma)^p u dx}{\sigma(Q)} \right)^{1/p} < \infty,$$

where $\sigma = v^{1-p'}$.

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where $\sigma = v^{1-p'}$.

- The S_p condition involves the operator under study itself.
- These testing conditions are defined for particular operators.

Sharp bounds for maximal functions

REMARK

The classical results were **qualitative results** since they **did not** reflect the quantitative dependence of the $L^p(w)$ operator norm in term of the relevant constant involving the weights.

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- **S. Buckley**

$$||M||_{L^p(w)} \leq C_p \, [w]_{A_p}^{\frac{1}{p-1}}.$$

- **J. Wittwer**: martingale operator and square function.
- **K. Moen**

$$||M||_{L^p(v) \longrightarrow L^p(u)} \approx [u,v]_{S_p}.$$

Singular integral operators

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- **Hilbert transform**

$$Hf(x) = \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \frac{1}{x-y} f(y)dy, \quad x \in \mathbb{R}$$

- **Riesz transforms**

$$R_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\, p.v. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y)dy, \qquad 1 \le j \le n.$$

- **Ahlfors-Beurling transform**

$$Bf(z) = p.v. \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} dw.$$

Calderón–Zygmund operators

DEFINITION

A linear operator T is a Calderón–Zygmund operator (CZO) if it extends to a bounded operator from $L^2(\mathbb{R}^n)$ into itself and there exists a function K defined off the diagonal of $x = y$, such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin supp\,(f), f \in C_c^\infty.$$

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- For certain $\delta > 0$,

$$|K(x,y) - K(x',y)| \lesssim \frac{|x-x'|^\delta}{(|x-y| + |x'-y|)^{n+\delta}},$$

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Sharp bounds for singular integral operators

- **S. Petermichl and A. Volberg**: Linear bound for the Beurling operator when $p \geq 2$.
- **S. Petermichl**: Optimal bounds for Hilbert and Riesz transforms.
- **O. Beznosova**: Linear bound for discrete paraproduct operators.
- **M. Lacey, S. Petermichl and M.C. Reguera**: Sharp A_2 bound for a large family of Haar shift operators.
- **D. Cruz-Uribe, J.M. Martell and C. Pérez**: More flexible method avoiding Bellman functions and two-weighted norm inequalities.
- **T. Hytönen**: Sharp A_2 bound for CZO (probabilistic approach).
- **A.K. Lerner**: Sharp A_2 bound for CZO (sparse operators).
- **K. Moen**: Sharp weighted bounds for sparse operators without extrapolation.

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$$[w]^H_{A_\infty} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(t)dt \right) exp \left(\frac{1}{|Q|} \int_Q logw(t)^{-1} dt \right)$$

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- **Starting point** for proving analogue results for other operators, i.e.,

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- Introduce the two-weight problem for $\mathscr{M}$.
- Prove the sharp bounds for the m-sparse operators.
- Prove some auxiliary results.