Spring 2016

Weighted inequalities for multilinear singular integrals

Unweighted inequalities with precise constants for bilinear Calderón–Zymgund operators of Dini type

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Bilinear Dini-type CZOs

Definition

T is a *ω-bilinear Calderón–Zygmund operator* if it is a bilinear operator originally defined as

$$T : \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n),$$

and for some $1 \leq q_1, q_2 < \infty$ it extends to a bounded bilinear operator from $L^{q_1} \times L^{q_2}$ to L^q, where $1/q_1 + 1/q_2 = 1/q$, and if there exists a function K, defined off the diagonal $x = y = z$ in $(\mathbb{R}^n)^3$, satisfying

$$T(f_1, f_2)(x) = \iint_{(\mathbb{R}^n)^2} K(x, y, z) f_1(y) f_2(z) dydz,$$

for all $x \notin supp f_1 \cap supp f_2$.

Bilinear Dini-type CZOs

DEFINITION

The kernel K must also satisfy, for some constants $C_K > 0$ and $\tau \in (0,1)$, the following size condition

$$|K(x,y,z)| \leq \frac{C_K}{(|x-y|+|x-z|)^{2n}},$$

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and, the smoothness estimate

$$\begin{aligned}
&|K(x+h,y,z)-K(x,y,z)|+|K(x,y+h,z)-K(x,y,z)| \\
&\quad +|K(x,y,z+h)-K(x,y,z)| \\
&\quad \leq \frac{1}{(|x-y|+|x-z|)^{2n}}\,\omega\left(\frac{|h|}{|x-y|+|x-z|}\right),
\end{aligned}$$

whenever $|h| \leq \tau \max(|x-y|,|x-z|)$.

Bilinear Dini-type CZOs

Here the function $\omega : [0,\infty) \to [0,\infty)$ will denote a modulus of continuity, that is,

- ω is an increasing function,
- ω is a subadditive function, i.e. $\omega(t+s) \leq \omega(t) + \omega(s)$, and
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Definition

K is a *Dini*(a)-continuous kernel if ω satisfies the following condition:

$$||\omega||_{\mathrm{Dini(a)}} := \int_0^1 \omega^a(t)\,\frac{dt}{t} < \infty.$$

In the case $a = 1$, we will denote $||\omega||_{\mathrm{Dini(a)}}$ simply as $||\omega||_{\mathrm{Dini}}$.

Weak inequality for a bilinear Dini-type CZO

LEMMA

Let T be a bilinear Dini-continuous Calderón-Zygmund operator. Then T is bounded from $L^1 \times L^1$ to $L^{\frac{1}{2},\infty}$ and

$$\|T\|_{L^1 \times L^1 \to L^{1/2,\infty}} \lesssim \|T\|_{L^{q_1} \times L^{q_2} \to L^q} + \|\omega\|_{\text{Dini}},$$

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- This result was proved under the stronger $Dini(\frac{1}{2})$ condition in [MN].

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- In [PT], the same result is proved under the bilinear geometric Hörmander condition (BGHC), which is weaker than the Dini condition.

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Lemma

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- This result was proved under the stronger $Dini(\frac{1}{2})$ condition in [MN].
- In [PT], the same result is proved under the bilinear geometric Hörmander condition (BGHC), which is weaker than the Dini condition.
- The precise constants were not considered in the previous proofs.

Proof of weak inequality

- Assume $T : L^{q_1} \times L^{q_2} \to L^q$ and WLOG that $f_i \geq 0$, $i = 1, 2$.

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- Apply the Calderón-Zygmund decomposition to f_i at height $\alpha_i \lambda$.
- We obtain good and bad parts g_i and b_i, respectively, and families of cubes $\{Q_k^i\}_k$ with disjoint interiors such that $f_i = g_i + b_i$ and $b_i = \sum_k b_k^i$ verifying the corresponding properties.

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- Set $\Omega_i = \cup_k 4nQ_k^i$.

We have

$$
\begin{aligned}
\big|\{x : |T(f_1,f_2)(x)| > \lambda\}\big| \leq |\Omega_1| + |\Omega_2| \\
&+ \Big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(g_1,g_2)(x)| > \frac{\lambda}{4}\}\Big| \\
&+ \Big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(g_1,b_2)(x)| > \frac{\lambda}{4}\}\Big| \\
&+ \Big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(b_1,g_2)(x)| > \frac{\lambda}{4}\}\Big| \\
&+ \Big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(b_1,b_2)(x)| > \frac{\lambda}{4}\}\Big|.
\end{aligned}
$$

First and second term. It is easy to see that

$$|\Omega_1| + |\Omega_2| \leq C_n\left(\frac{1}{\alpha_1\lambda}\|f_1\|_{L^1} + \frac{1}{\alpha_2\lambda}\|f_2\|_{L^1}\right).$$

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Third term. Using Chebychev's inequality and the boundedness properties of T and g_i, we have

$$\begin{aligned}
&\Big|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(g_1, g_2)(x)| > \frac{\lambda}{4}\}\Big| \\
&\leq \frac{4^q}{\lambda^q} \|T(g_1, g_2)\|_{L^q}^q \\
&\leq \frac{4^q}{\lambda^q} \|T\|_{L^{q_1} \times L^{q_2} \to L^q}^q \|g_1\|_{L^{q_1}}^q \|g_2\|_{L^{q_2}}^q \\
&\leq \frac{4^q}{\lambda^q} C_{n,q,q_1,q_2} \|T\|_{L^{q_1} \times L^{q_2} \to L^q}^q (\alpha_1 \lambda)^{q/q_1'} (\alpha_2 \lambda)^{q/q_2'} \|f_1\|_{L^1}^{q/q_1} \|f_2\|_{L^1}^{q/q_2}.
\end{aligned}$$

Fourth and fifth* term. Denote the center of the cube Q_k^2 by c_k. We get

$$
\begin{aligned}
&\Big|\{x\in(\Omega_1\cup\Omega_2)^c : |T(g_1,b_2)(x)|>\frac{\lambda}{4}\}\Big| \\
\leq\ & \frac{4}{\lambda}\int\Big|\sum_k\int\int_{Q_k^2}(K(x,y,z)-K(x,y,c_k))g_1(y)b_2^k(z)dzdy\Big|dx \\
\leq\ & \frac{4}{\lambda}\sum_k\int\int\int_{Q_k^2}|K(x,y,z)-K(x,y,c_k)|\cdot|g_1(y)|\cdot|b_2^k(z)|dzdydx \\
\leq\ & \frac{4}{\lambda}\sum_k\int_{Q_k^2}\int\int\omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2(|x-y|+|x-z|)}\Big)\frac{|g_1(y)|\cdot|b_2^k(z)|}{(|x-y|+|x-z|)^{2n}}dydxdz \\
\leq\ & C_n\alpha_1\sum_k\int_{Q_k^2}\int\int\omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2(|y|+|x-z|)}\Big)\frac{|b_2^k(z)|}{(|y|+|x-z|)^{2n}}dydxdz \\
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\leq\ & C_n\alpha_1\sum_k\int_{Q_k^2}\int_{|x-z|\geq n\ell(Q_k^2)}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2)}{2|x-z|}\Big)\frac{|b_2^k(z)|}{|x-z|^n}dxdz \\
\leq\ & C_n'\alpha_1\|\omega\|_{\text{Dini}}\|f_2\|_{L^1}.
\end{aligned}
$$

Last term. Denote by c_l and c_k the center of the cubes Q_l^1 and Q_k^2, respectively. We obtain,

$$\begin{aligned} & \left|\{x \in (\Omega_1 \cup \Omega_2)^c : |T(b_1, b_2)(x)| > \frac{\lambda}{4}\}\right| \\ \leq \quad & \frac{4}{\lambda} \int \left| \sum_{k,l} \int_{Q_l^1} \int_{Q_k^2} (K(x,y,z) - K(x,y,c_k)) b_1^l(y) b_2^k(z) dz dy \right| dx \end{aligned}$$

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\leq \quad & \frac{C_n}{\lambda} \sum_{k,l} \int_{Q_k^2} \int_{Q_l^1} \int_{(\Omega_1 \cup \Omega_2)^c} \omega\left(\frac{\sqrt{n}\ell(Q_k^2)}{2(|x-c_l| + |x-c_k|)}\right) \frac{|b_1^l(y)| |b_2^k(z)| dx dy dz}{(|x-c_l| + |x-c_k|)^{2n}}
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\leq \quad & C_n \sum_{k,l} |Q_l^1| |Q_k^2| \alpha_1 \alpha_2 \lambda \int_{(\Omega_1 \cup \Omega_2)^c} \omega\left(\frac{\sqrt{n}(\ell(Q_k^2) + \ell(Q_l^1))}{2(|x-c_l|+|x-c_k|)}\right) \frac{dx}{(|x-c_l|+|x-c_k|)^{2n}}
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\leq\ & C_n\sum_{k,l}|Q_l^1||Q_k^2|\alpha_1\alpha_2\lambda\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}(\ell(Q_k^2)+\ell(Q_l^1))}{2(|x-c_l|+|x-c_k|)}\Big)\frac{dx}{(|x-c_l|+|x-c_k|)^{2n}} \\
\leq\ & C_n'\sum_{k,l}\alpha_1\alpha_2\lambda\int_{Q_k^2}\int_{Q_l^1}\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}(\ell(Q_k^2)+\ell(Q_l^1))}{2(|x-y|+|x-z|)}\Big)\frac{dxdydz}{(|x-y|+|x-z|)^{2n}}
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\leq \quad & C_n' \sum_{k,l} \alpha_1\alpha_2\lambda \Big(\int_{\ell(Q_k^2)\geq \ell(Q_l^1)} + \int_{\ell(Q_l^1)\geq \ell(Q_k^2)}\Big) \\
\leq \quad & I+II.
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\begin{aligned}
&\Big|\{\quad x\in\quad(\Omega_1\cup\Omega_2)^c:|T(b_1,b_2)(x)|>\frac{\lambda}{4}\}\Big|\\
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&\quad\leq\quad C_n'\sum_{k,l}\alpha_1\alpha_2\lambda\Big(\int_{\ell(Q_k^2)\geq\ell(Q_l^1)}+\int_{\ell(Q_l^1)\geq\ell(Q_k^2)}\Big)\\
&\quad\leq\quad I+II.
\end{aligned}
$$

By symmetry, it suffices to estimate I. We have

$$
\begin{aligned}
I\quad&\leq\quad C_n'\sum_{k}\alpha_1\alpha_2\lambda\int_{Q_k^2}\int_{(\Omega_1\cup\Omega_2)^c}\int_{\mathbb{R}^n}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2))}{|x-z|}\Big)\frac{dydxdz}{(|x-y|+|x-z|)^{2n}}\\
&=\quad C_n'\sum_{k}\alpha_1\alpha_2\lambda\int_{Q_k^2}\int_{(\Omega_1\cup\Omega_2)^c}\omega\Big(\frac{\sqrt{n}\ell(Q_k^2))}{|x-z|}\Big)\frac{1}{|x-z|^n}dxdz\\
&\leq\quad C_n\alpha_1\|\omega\|_{\text{Dini}}\|f_2\|_{L^1}.
\end{aligned}
$$

Combining the arguments above, we have

$$\begin{aligned}
&\big|\{x : |T(f_1,f_2)(x)| > \lambda\}\big| \\
&\qquad\lesssim \frac{1}{\alpha_1\lambda}\|f_1\|_{L^1} + \frac{1}{\alpha_2\lambda}\|f_2\|_{L^1} \\
&\qquad+ \|T\|^q_{L^{q_1}\times L^{q_2}\to L^q}(\alpha_1)^{q/q_1'}(\alpha_2)^{q/q_2'}\lambda^{q-1}\|f_1\|_{L^1}^{q/q_1}\|f_2\|_{L^1}^{q/q_2} \\
&\qquad+ \alpha_1\|\omega\|_{\text{Dini}}\|f_2\|_{L^1} + \alpha_2\|\omega\|_{\text{Dini}}\|f_1\|_{L^1}
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&\left|\{x : |T(f_1,f_2)(x)| > \lambda\}\right| \\
&\quad \lesssim \frac{1}{\alpha_1\lambda}\|f_1\|_{L^1} + \frac{1}{\alpha_2\lambda}\|f_2\|_{L^1} \\
&\quad + \|T\|^q_{L^{q_1}\times L^{q_2}\to L^q}(\alpha_1)^{q/q_1'}(\alpha_2)^{q/q_2'}\lambda^{q-1}\|f_1\|^{q/q_1}_{L^1}\|f_2\|^{q/q_2}_{L^1} \\
&\quad + \alpha_1\|\omega\|_{\text{Dini}}\|f_2\|_{L^1} + \alpha_2\|\omega\|_{\text{Dini}}\|f_1\|_{L^1}
\end{aligned}
$$

Thus, taking

$$
\alpha_1 = \lambda^{-\frac{1}{2}}\frac{\|f_1\|^{\frac{1}{2}}_{L^1}}{\|f_2\|^{\frac{1}{2}}_{L^1}}\frac{1}{(\|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\text{Dini}})^{\frac{1}{2}}}
$$

$$
\alpha_2 = \lambda^{-\frac{1}{2}}\frac{\|f_2\|^{\frac{1}{2}}_{L^1}}{\|f_1\|^{\frac{1}{2}}_{L^1}}\frac{1}{(\|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\text{Dini}})^{\frac{1}{2}}},
$$

we get

$$
\lambda\left|\{x : |T(f_1,f_2)(x)| > \lambda\}\right|^2 \le (\|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\text{Dini}})\|f_1\|_{L^1}\|f_2\|_{L^1}.
$$

□

Maximal truncation of a bilinear CZO

Definition

Given a bilinear Calderón-Zygmund operator T, the maximal truncation of T is defined as the operator $T_\sharp$ given by

$$T_\sharp(f_1,f_2)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f_1,f_2)(x)|,$$

where T_ε is the ε-truncation of T

$$T_\varepsilon(f_1,f_2)(x) = \int_{|x-y|^2+|x-z|^2>\varepsilon^2} K(x,y,z)f_1(y)f_2(z)dydz.$$

Cotlar's inequality for $T^\sharp$

LEMMA

Let T be a bilinear Dini-continuous Calderón-Zygmund operator with kernel K. Then, for all $\eta \in (0, \frac{1}{2})$, there exists a constant C such that

$$T_\sharp(\vec{f}) \leq c_{\eta,n}(C_K + ||\omega||_{\text{Dini}} + ||T||_{L^{q_1}\times L^{q_2}\to L^q})\mathscr{M}(\vec{f}) + M_\eta(|T(\vec{f})|),$$

where

$$M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}, \quad \eta > 0.$$

Proof

Let us begin defining the following maximal truncation

$$\widetilde{T}_{\sharp}(f_1,f_2)(x) = \sup_{\varepsilon>0} \left| \widetilde{T_{\varepsilon}}(f_1,f_2)(x) \right|,$$

where

$$\widetilde{T_{\varepsilon}}(f_1,f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} K(x,y,z) f_1(y) f_2(z)\, dy dz.$$

Proof

Let us begin defining the following maximal truncation

$$\widetilde{T}_\sharp(f_1,f_2)(x) = \sup_{\varepsilon>0}\left|\widetilde{T_\varepsilon}(f_1,f_2)(x)\right|,$$

where

$$\widetilde{T_\varepsilon}(f_1,f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} K(x,y,z)f_1(y)f_2(z)dydz.$$

Since

$$\sup_{\varepsilon>0}\left|\int_{\substack{\max\{|x-y|,|x-z|\}\leq\varepsilon \\ |x-y|^2+|x-z|^2>\varepsilon^2}} K(x,y,z)f_1(y)f_2(z)dydz\right| \lesssim C_K\mathscr{M}(f_1,f_2)(x),$$

it suffices to show the desired inequality with $T_\sharp$ replaced by $\widetilde{T}_\sharp$.

Proof

Notice that we can write for $x' \in B(x, \varepsilon/2)$,

$$\widetilde{T}_\varepsilon(f_1,f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x,y,z) - K(x',y,z))f_1(y)f_2(z)dydz$$
$$+ T(f_1,f_2)(x') - T(f_1^0,f_2^0)(x'),$$

where $f_i^0 = f_i \mathbf{1}_{B(x,\varepsilon)}$.

Proof

Notice that we can write for $x' \in B(x, \varepsilon/2)$,

$$\widetilde{T}_\varepsilon(f_1,f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x,y,z) - K(x',y,z))f_1(y)f_2(z)dydz$$
$$+T(f_1,f_2)(x') - T(f_1^0,f_2^0)(x'),$$

where $f_i^0 = f_i \mathbf{1}_{B(x,\varepsilon)}$.

$$\widetilde{T}_\sharp(f_1,f_2)(x) = \sup_{\varepsilon>0} \left|\widetilde{T_\varepsilon}(f_1,f_2)(x)\right|,$$

where

$$\widetilde{T_\varepsilon}(f_1,f_2)(x) = \int_{\max\{|x-y|,|x-z|\}>\varepsilon} K(x,y,z)f_1(y)f_2(z)dydz.$$

Proof

For the first term, using the regularity assumptions on the kernel, we get

$$\Big|\int_{\max\{|x-y|,|x-z|\}>\varepsilon}(K(x,y,z)-K(x',y,z))f_1(y)f_2(z)dydz\Big|$$

$$\leq \int_{\max\{|x-y|,|x-z|\}>\varepsilon}\omega\Big(\frac{|x-x'|}{|x-y|+|x-z|}\Big)\frac{|f_1(y)||f_2(z)|dydz}{(|x-y|+|x-z|)^{2n}}$$

Proof

For the first term, using the regularity assumptions on the kernel, we get

$$
\begin{aligned}
&\Big|\int_{\max\{|x-y|,|x-z|\}>\varepsilon}(K(x,y,z)-K(x',y,z))f_1(y)f_2(z)dydz\Big| \\
\leq\ & \int_{\max\{|x-y|,|x-z|\}>\varepsilon}\omega\left(\frac{|x-x'|}{|x-y|+|x-z|}\right)\frac{|f_1(y)||f_2(z)|dydz}{(|x-y|+|x-z|)^{2n}} \\
=\ & \sum_{k=0}^{\infty}\int_{2^k\varepsilon<\max\{|x-y|,|x-z|\}\leq 2^{k+1}\varepsilon}\omega\left(\frac{|x-x'|}{2^k\varepsilon}\right)\frac{1}{(2^k\varepsilon)^{2n}}|f_1(y)||f_2(z)|dydz
\end{aligned}
$$

Proof

For the first term, using the regularity assumptions on the kernel, we get

$$\begin{aligned}
&\left|\int_{\max\{|x-y|,|x-z|\}>\varepsilon}(K(x,y,z)-K(x',y,z))f_1(y)f_2(z)dydz\right| \\
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=\;& \sum_{k=0}^{\infty}\int_{2^k\varepsilon<\max\{|x-y|,|x-z|\}\leq 2^{k+1}\varepsilon}\omega\left(\frac{|x-x'|}{2^k\varepsilon}\right)\frac{1}{(2^k\varepsilon)^{2n}}|f_1(y)||f_2(z)|dydz \\
\lesssim\;& \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\omega\left(\frac{|x-x'|}{2^k\varepsilon}\right)
\end{aligned}$$

Proof

For the first term, using the regularity assumptions on the kernel, we get

$$
\begin{aligned}
& \Big| \int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x,y,z) - K(x',y,z)) f_1(y) f_2(z)\, dydz \Big| \\
\leq\ & \int_{\max\{|x-y|,|x-z|\}>\varepsilon} \omega\Big(\frac{|x-x'|}{|x-y|+|x-z|}\Big) \frac{|f_1(y)||f_2(z)|\,dydz}{(|x-y|+|x-z|)^{2n}} \\
=\ & \sum_{k=0}^{\infty} \int_{2^k\varepsilon<\max\{|x-y|,|x-z|\}\leq 2^{k+1}\varepsilon} \omega\Big(\frac{|x-x'|}{2^k\varepsilon}\Big) \frac{1}{(2^k\varepsilon)^{2n}} |f_1(y)||f_2(z)|\,dydz \\
\lesssim\ & \mathcal{M}(f_1,f_2)(x) \sum_{k=0}^{\infty} \omega\Big(\frac{|x-x'|}{2^k\varepsilon}\Big) \\
\lesssim\ & \mathcal{M}(f_1,f_2)(x) \sum_{k=0}^{\infty} \int_{2^{k-1}}^{2^k} \omega\big(\frac{|x-x'|}{\varepsilon t}\big) \frac{dt}{t}
\end{aligned}
$$

Proof

For the first term, using the regularity assumptions on the kernel, we get

$$
\begin{aligned}
&\Big| \int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x,y,z) - K(x',y,z)) f_1(y) f_2(z)\, dy dz \Big| \\
\leq \;& \int_{\max\{|x-y|,|x-z|\}>\varepsilon} \omega\Big(\frac{|x-x'|}{|x-y|+|x-z|} \Big) \frac{|f_1(y)||f_2(z)|\, dy dz}{(|x-y|+|x-z|)^{2n}} \\
= \;& \sum_{k=0}^{\infty} \int_{2^k\varepsilon<\max\{|x-y|,|x-z|\}\leq 2^{k+1}\varepsilon} \omega\Big(\frac{|x-x'|}{2^k\varepsilon} \Big) \frac{1}{(2^k\varepsilon)^{2n}} |f_1(y)||f_2(z)|\, dy dz \\
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\lesssim \;& \mathcal{M}(f_1,f_2)(x) \sum_{k=0}^{\infty} \int_{2^{k-1}}^{2^k} \omega\Big(\frac{|x-x'|}{\varepsilon t} \Big) \frac{dt}{t} \\
= \;& \mathcal{M}(f_1,f_2)(x) \sum_{k=0}^{\infty} \int_{\frac{|x-x'|}{2^k\varepsilon}}^{\frac{|x-x'|}{2^{k-1}\varepsilon}} \omega(u) \frac{du}{u}
\end{aligned}
$$

Proof

For the first term, using the regularity assumptions on the kernel, we get

$$
\begin{aligned}
&\left|\int_{\max\{|x-y|,|x-z|\}>\varepsilon} (K(x,y,z)-K(x',y,z))f_1(y)f_2(z)dydz\right| \\
\leq\ & \int_{\max\{|x-y|,|x-z|\}>\varepsilon} \omega\left(\frac{|x-x'|}{|x-y|+|x-z|}\right)\frac{|f_1(y)||f_2(z)|dydz}{(|x-y|+|x-z|)^{2n}} \\
=\ & \sum_{k=0}^{\infty}\int_{2^k\varepsilon<\max\{|x-y|,|x-z|\}\leq 2^{k+1}\varepsilon} \omega\left(\frac{|x-x'|}{2^k\varepsilon}\right)\frac{1}{(2^k\varepsilon)^{2n}}|f_1(y)||f_2(z)|dydz \\
\lesssim\ & \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\omega\left(\frac{|x-x'|}{2^k\varepsilon}\right) \\
\lesssim\ & \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\int_{2^{k-1}}^{2^k}\omega\left(\frac{|x-x'|}{\varepsilon t}\right)\frac{dt}{t} \\
=\ & \mathcal{M}(f_1,f_2)(x)\sum_{k=0}^{\infty}\int_{\frac{|x-x'|}{2^k\varepsilon}}^{\frac{|x-x'|}{2^{k-1}\varepsilon}}\omega(u)\frac{du}{u} \\
=\ & \mathcal{M}(f_1,f_2)(x)\int_0^{\frac{2|x-x'|}{\varepsilon}}\omega(u)\frac{du}{u} \leq \|\omega\|_{\mathrm{Dini}}\mathcal{M}(f_1,f_2)(x),
\end{aligned}
$$

where the last step holds since $|x-x'|\leq\varepsilon/2$.

Proof

Next, taking the L^η average over $x' \in B(x, \varepsilon/2)$, we arrive at

$$|\widetilde{T_\varepsilon}(f_1,f_2)(x)| \lesssim ||\omega||_{\text{Dini}}\mathscr{M}(f_1,f_2)(x) + M_\eta(|T(f_1,f_2)|)(x) + \left(\frac{1}{|B(x,\varepsilon/2)|}\int_{B(x,\varepsilon/2)} |T(f_1^0,f_2^0)(x')|^\eta dx'\right)^{1/\eta}.$$

Proof

Next, taking the L^η average over $x' \in B(x,\varepsilon/2)$, we arrive at

$$\begin{aligned}|\widetilde{T_\varepsilon}(f_1,f_2)(x)| &\lesssim ||\omega||_{\text{Dini}}\mathscr{M}(f_1,f_2)(x) + M_\eta(|T(f_1,f_2)|)(x)\\ &+ \left(\frac{1}{|B(x,\varepsilon/2)|}\int_{B(x,\varepsilon/2)}|T(f_1^0,f_2^0)(x')|^\eta dx'\right)^{1/\eta}.\end{aligned}$$

For the last term, using Kolmogorov's inequality, we obtain for any $\eta \in (0,\frac{1}{2})$,

$$\begin{aligned}&\left(\frac{1}{|B(x,\varepsilon/2)|}\int_{B(x,\varepsilon/2)}|T(f_1^0,f_2^0)(x')|^\eta dx'\right)^{1/\eta}\\ &= ||T(f_1^0,f_2^0)||_{L^\eta(B(x,\frac{\varepsilon}{2}),\frac{dx}{|B(x,\frac{\varepsilon}{2})|})}\\ &\leq C_\eta ||T(f_1^0,f_2^0)||_{L^{1/2,\infty}(B(x,\frac{\varepsilon}{2}),\frac{dx}{|B(x,\frac{\varepsilon}{2})|})}\\ &\leq C_\eta ||T||_{L^1\times L^1\to L^{1/2,\infty}}\mathscr{M}(f_1,f_2)(x).\end{aligned}$$

Proof

Combining all the terms, we finally arrive at

$$|\widetilde{T}_\varepsilon(f_1,f_2)(x)| \le c_n(||\omega||_{\text{Dini}} + C_\eta ||T||_{L^1\times L^1\to L^{1/2,\infty}})\mathscr{M}(f_1,f_2)(x) + M_\eta(|T(f_1,f_2)|)(x),$$

Proof

Combining all the terms, we finally arrive at

$$|\widetilde{T}_\varepsilon(f_1,f_2)(x)| \leq c_n(||\omega||_{\text{Dini}} + C_\eta ||T||_{L^1\times L^1\to L^{1/2,\infty}})\mathscr{M}(f_1,f_2)(x) \\ + M_\eta(|T(f_1,f_2)|)(x),$$

which taking into account the relationship between $T^\sharp$ and $\tilde{T}^\sharp$ and that

$$\|T\|_{L^1\times L^1\to L^{1/2,\infty}} \lesssim \|T\|_{L^{q_1}\times L^{q_2}\to L^q} + \|\omega\|_{\text{Dini}},$$

leads to the desired result.

□

Weak inequality for the maximal truncation

Corollary

Let T be a bilinear Calderón–Zygmund operator with Dini-continuous kernel K. Then

$$||T_\sharp||_{L^1\times L^1\to L^{1/2,\infty}} \lesssim (C_K + ||\omega||_{\mathrm{Dini}} + ||T||_{L^{q_1}\times L^{q_2}\to L^q}).$$

Weak inequality for the maximal truncation

COROLLARY

Let T be a bilinear Calderón–Zygmund operator with Dini-continuous kernel K. Then

$$||T_\sharp||_{L^1\times L^1\to L^{1/2,\infty}} \lesssim (C_K + ||\omega||_{\text{Dini}} + ||T||_{L^{q_1}\times L^{q_2}\to L^q}).$$

We need to recall the definition of the *adjacent dyadic systems* $\mathscr{D}_u$,

$$\mathscr{D}^u := \{2^{-k}([0,1)^u + m + (-1)^k \tfrac{1}{3}u) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad u \in \{0,1,2\}^n.$$

LEMMA (HYTÖNEN-RONCAL-TAPIOLA, 2015)

If $Q_0 \in \cup_{u\in\{0,1,2\}^n}\mathscr{D}^u$, then for any ball $B := B(x,r) \subset Q_0$ there exists a cube $Q_B \in \cup_{u\in\{0,1,2\}^n}\mathscr{D}^u$ such that $B \subset Q_B \subseteq Q_0$ and $\ell(Q_B) \leq 12r$.

Proof

Fix $\eta \in (0,1/2)$. Note that Cotlar's inequality together with the weak boundedness of the multilinear maximal function and bilinear Calderón–Zygmund operators and the fact that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$ gives us the desired result.

Proof

Fix $\eta \in (0,1/2)$. Note that Cotlar's inequality together with the weak boundedness of the multilinear maximal function and bilinear Calderón–Zygmund operators and the fact that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$ gives us the desired result.

It is enough to prove that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$.

Proof

Fix $\eta \in (0,1/2)$. Note that Cotlar's inequality together with the weak boundedness of the multilinear maximal function and bilinear Calderón–Zygmund operators and the fact that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$ gives us the desired result.

It is enough to prove that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$. Indeed,

$$M(f) \approx \sum_{u=1}^{3^n} M_u(f),$$

where

$$M_u(f) := \sup_{\substack{Q \ni x \\ Q \in \mathscr{D}^u}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Proof

Fix $\eta \in (0,1/2)$. Note that Cotlar's inequality together with the weak boundedness of the multilinear maximal function and bilinear Calderón–Zygmund operators and the fact that $M_\eta \circ T : L^1 \times L^1 \to L^{1/2,\infty}$ gives us the desired result.

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where

$$M_u(f) := \sup_{\substack{Q \ni x \\ Q \in \mathscr{D}^u}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Therefore,

$$\Big|\{x : M(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda\}\Big| \leq \sum_{u=1}^{3^n} \Big|\{x : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}\Big|.$$

Proof

Denote

$$E_u := \{x \in \mathbb{R}^n : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}.$$

Proof

Denote

$$E_u := \{x \in \mathbb{R}^n : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}.$$

We can find a collection of maximal dyadic cubes $\{Q_j\}_j$ such that $E_u = \cup_j Q_j$ and

$$\frac{1}{|Q_j|}\int_{Q_j} |T(f_1,f_2)|^\eta > \lambda^\eta (3^n)^{-\eta},$$

Proof

Denote

$$E_u := \{x \in \mathbb{R}^n : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}.$$

We can find a collection of maximal dyadic cubes $\{Q_j\}_j$ such that $E_u = \cup_j Q_j$ and

$$\frac{1}{|Q_j|}\int_{Q_j} |T(f_1,f_2)|^\eta > \lambda^\eta (3^n)^{-\eta},$$

which means that

$$|E_u| \leq (3^n)^\eta \lambda^{-\eta} \int_{E_u} |T(f_1,f_2)|^\eta, \quad u = 1,\ldots,3^n.$$

Proof

Denote

$$E_u := \{x \in \mathbb{R}^n : M_u(|T(f_1,f_2)|^\eta)(x)^{\frac{1}{\eta}} > \lambda/3^n\}.$$

We can find a collection of maximal dyadic cubes $\{Q_j\}_j$ such that $E_u = \cup_j Q_j$ and

$$\frac{1}{|Q_j|}\int_{Q_j} |T(f_1,f_2)|^\eta > \lambda^\eta (3^n)^{-\eta},$$

which means that

$$|E_u| \leq (3^n)^\eta \lambda^{-\eta} \int_{E_u} |T(f_1,f_2)|^\eta, \quad u = 1,\dots,3^n.$$

Now using Kolmogorov's inequality and the fact that $T : L^1 \times L^1 \to L^{1/2,\infty}$, and assuming that $\eta < 1/2$, we get

$$\int_{E_u} |T(f_1,f_2)|^\eta \lesssim \|T(f_1,f_2)\|^\eta_{L^{\frac{1}{2},\infty}(E_u,\frac{dx}{|E_u|})} |E_u| \leq \|f_1\|_1^\eta \|f_2\|_1^\eta |E_u|^{1-2\eta}$$

Proof

Combining both estimates, it follows that

$$|E_u| \leq \lambda^{-\eta}(3^n)^\eta \|f_1\|_1^\eta \|f_2\|_1^\eta |E_u|^{1-2\eta},$$

which is exactly,

$$\lambda |E_u|^2 \leq c_{n,\eta} \|f_1\|_1 \|f_2\|_1.$$

□