Spring 2016

Weighted inequalities for multilinear singular integrals

Characterization of $A_{\vec{P}}$ weights

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21 March 2016

Weight constants

A_p CONDITION

Let $1 < p < \infty$. A weight w satisfies the A_p condition if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} < \infty.$$

$A_{\vec{P}}$ CONDITION

Let $1 \leq p_1, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. A vector of weights $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}\right)\prod_{j=1}^{m}\left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{p/p_j'} < \infty.$$

where,

$$v_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}.$$

Characterization of $A_{\vec{P}}$ classes

PROPOSITION [LOPTT]

Let $\vec{w} = (w_1, \cdots, w_m)$ and $1 \leq p_1, \ldots, p_m < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if and only if

$$\begin{cases} w_j^{1-p_j'} \in A_{mp_j'},\ j = 1, \ldots, m \\ \nu_{\vec{w}} \in A_{mp}, \end{cases} \tag{B}$$

where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ must be understood as $w_j^{1/m} \in A_1$.

Proof

We study two cases:

- *Case 1:* At least one $p_j > 1$.

$$\vec{w} \in A_{\vec{P}} \Leftrightarrow \begin{cases} w_j^{1-p'_j} \in A_{mp'_j}, \ p_j > 1 \\ \nu_{\vec{w}} \in A_{mp} \\ w_j^{1/m} \in A_1, \ p_j = 1 \end{cases}$$

- *Case 2:* All $p_j = 1, j = 1, \ldots, m$.

$$\vec{w} \in A_{(1,\ldots,1)} \Leftrightarrow \begin{cases} \nu_{\vec{w}} \in A_1 \\ w_j^{1/m} \in A_1, \ j = 1, \ldots, m. \end{cases}$$

Proof: Case 1

CASE 1: At least one $p_j > 1$

Assume WLOG that

- $p_1, \ldots, p_l = 1, \qquad 0 \leq l < m,$
- $p_j > 1, \qquad j = l+1, \ldots, m.$

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Assume WLOG that

- $p_1, \ldots, p_l = 1, \qquad 0 \leq l < m,$
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$\boxed{A_{\vec{P}} \Rightarrow \text{(B)}}$

Fix $j \geq l+1$ and define the numbers

$$\begin{cases} q_j = p\left(m - 1 + \frac{1}{p_j}\right) \\ q_i = \frac{p_i}{p_i - 1}\frac{q_j}{p}, & i \neq j,\ i \geq l+1. \end{cases}$$

Proof: Case 1

We want to show

$$w_j^{1-p_j'} \in A_{mp_j'} \Leftrightarrow \left(\int_Q w_j^{1-p_j'}\right)\left(\int_Q w_j^{(1-p_j')(1-(mp_j')')}\right)^{mp_j'-1} \leq C|Q|^{mp_j'}$$

$$\Leftrightarrow \left(\int_Q w_j^{1-p_j'}\right)\left(\int_Q w_j^{\frac{p}{p_j q_j}}\right)^{\frac{q_j p_j}{p(p_j-1)}} \leq C|Q|^{\frac{mp_j}{p_j-1}}$$

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Indeed,

$$\frac{q_j p_j}{p(p_j-1)} = \frac{p(m-1+\frac{1}{p_j})}{p} p_j' = mp_j' - p_j' + \frac{p_j'}{p_j} = mp_j' - p_j' - (1-p_j') = mp_j' - 1$$

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$$w_j^{1-p_j'} \in A_{mp_j'} \Leftrightarrow \left(\int_Q w_j^{1-p_j'}\right)\left(\int_Q w_j^{(1-p_j')(1-(mp_j')')}\right)^{mp_j'-1} \leq C|Q|^{mp_j'}$$

$$\Leftrightarrow \left(\int_Q w_j^{1-p_j'}\right)\left(\int_Q w_j^{\frac{p}{p_jq_j}}\right)^{\frac{q_jp_j}{p(p_j-1)}} \leq C|Q|^{\frac{mp_j}{p_j-1}}$$

Indeed,

$$(1-p_j')(1-(mp_j')') = (1-p_j')\frac{1}{1-mp_j'} = \frac{1}{(1-p_j)(1-mp_j')} = \frac{1}{mp_j-p_j+1}$$

$$\frac{p}{p_jq_j} = \frac{p}{p_jp(m-1+1/p_j)} = \frac{1}{mp_j-p_j+1}$$

Proof: Case 1

$$\sum_{i=l+1}^{m} \frac{1}{q_i} = \frac{1}{q_j} + \sum_{\substack{i=l+1 \\ i \neq j}}^{m} \frac{p}{p'_i} \frac{1}{q_j}$$

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$$= \frac{1}{q_j}(1 + p \sum_{\substack{i=l+1 \\ i\neq j}}^{m} \frac{1}{p'_i})$$

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&= \frac{1}{q_j}(1 + p \sum_{\substack{i=l+1 \\ i\neq j}}^{m} \frac{1}{p'_i}) \\
&= \frac{1}{q_j}(1 + p \sum_{\substack{i=l+1 \\ i\neq j}}^{m} 1 - \frac{1}{p_i})
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&= \frac{1}{q_j}(1 + p \sum_{\substack{i=l+1\\ i\neq j}}^{m} 1 - \frac{1}{p_i}) \\
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&= \frac{1}{q_j}(1 + p(m - l - 1 - \sum_{\substack{i=l+1\\ i\neq j}}^{m} \frac{1}{p_i})) \\
&= \frac{1}{q_j}(1 + p(m - l - 1 - \frac{1}{p} + \frac{1}{p_j} + l))
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&= \frac{1}{q_j}(1 + p(m - l - 1 - \frac{1}{p} + \frac{1}{p_j} + l)) \\
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&= \frac{1}{q_j}(1 + p(m - l - 1 - \frac{1}{p} + \frac{1}{p_j} + l)) \\
&= \frac{pm - p + \frac{p}{p_j}}{q_j} = 1.
\end{aligned}$$

Proof: Case 1

Applying Hölder's inequality, we obtain

$$\int_Q w_j^{\frac{p}{p_j q_j}} = \int_Q \left(\prod_{i=l+1}^{m} w_i^{\frac{p}{p_i q_j}}\right)\left(\prod_{\substack{i=l+1\\ i\neq j}}^{m} w_i^{\frac{-p}{p_i q_j}}\right)$$

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&\leq \left(\int_Q \prod_{i=l+1}^{m} w_i^{p/p_i} \right)^{1/q_j} \prod_{\substack{i=l+1 \\ i \neq j}}^{m} \left(\int_Q w_i^{\frac{-p}{p_i q_j} q_i} \right)^{1/q_i} \\
&= \left(\int_Q \prod_{i=l+1}^{m} w_i^{p/p_i} \right)^{1/q_j} \prod_{\substack{i=l+1 \\ i \neq j}}^{m} \left(\int_Q w_i^{\frac{-1}{p_i - 1}} \right)^{1/q_i} .
\end{aligned}
$$

Proof: Case 1

From this inequality and the $A_{\vec{P}}$ condition follows

$$\left(\int_Q w_j^{1-p_j'}\right)\left(\int_Q w_j^{\frac{p}{p_j q_j}}\right)^{\frac{q_j p_j}{p(p_j-1)}}$$

$$\leq \left(\int_Q w_j^{1-p_j'}\right)\left(\left(\int_Q \prod_{i=l+1}^{m} w_i^{p/p_i}\right)^{1/q_j} \prod_{\substack{i=l+1\\ i\neq j}}^{m} \left(\int_Q w_i^{1-p_i'}\right)^{1/q_i}\right)^{\frac{q_j p_j}{p(p_j-1)}}$$

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\leq\ &\left(\int_Q \prod_{i=l+1}^{m} w_i^{p/p_i}\right)^{\frac{p_j'}{p}} \prod_{\substack{i=l+1\\ i\neq j}}^{m} \left(\int_Q w_i^{1-p_i'}\right)^{\frac{q_j}{q_i}\frac{p_j'}{p}} \left(\int_Q w_j^{1-p_j'}\right)
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$$\leq \left(\int_Q \prod_{i=l+1}^{m} w_i^{p/p_i}\right)^{\frac{p_j'}{p}} \prod_{\substack{i=l+1\\ i\neq j}}^{m}\left(\int_Q w_i^{1-p_i'}\right)^{\frac{q_j}{q_i}\frac{p_j'}{p}} \left(\int_Q w_j^{1-p_j'}\right)$$

$$\leq \left(\int_Q \prod_{i=1}^{m} w_i^{p/p_i} \prod_{i=1}^{l}(ess\ inf_Q w_i)^{-p/p_i}\right)^{\frac{p_j'}{p}} \prod_{i=l+1}^{m}\left(\int_Q w_i^{1-p_i'}\right)^{\frac{p}{p_i'}\frac{p_j'}{p}}$$

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$$s_j = (m - \frac{1}{p})p'_j, \qquad j \geq l+1$$

we have

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we have

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Using Hölder's inequality, we obtain

$$\int_Q \prod_{j=l+1}^{m} w_j^{-\frac{p}{p_j(pm-1)}} \leq \prod_{j=l+1}^{m} \left(\int_Q w_j^{-1/(p_j-1)} \right)^{1/s_j}.$$

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Hence, using the definition of $\nu_{\vec{w}}$ and the previous inequality

$$\int_Q (\nu_{\vec{w}})^{-\frac{1}{pm-1}} \leq \prod_{j=1}^{l} (ess\, inf_Q\, w_j)^{-\frac{p}{pm-1}} \prod_{j=l+1}^{m} \left(\int_Q w_j^{-1/(p_j-1)} \right)^{1/s_j}.$$

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Combining this inequality and the $A_{\vec{P}}$ condition gives that $\nu_{\vec{w}} \in A_{mp}$. Indeed,

$$\left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}^{1-(mp)'} \right)^{mp-1}$$

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$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right)\left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{1-(mp)'}\right)^{mp-1} \\ \leq\ &\left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right)\prod_{j=1}^l (ess\, inf_Q\, w_j)^{-\frac{p}{mp-1}(mp-1)} \prod_{j=l+1}^m \left(\frac{1}{|Q|}\int_Q w_j^{\frac{-1}{p_j-1}}\right)^{p/p_j'} < C \end{aligned}$$

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Fix $1 \le i_0 \le l$. By Hölder's inequality and

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we obtain

$$\int_Q w_{i_0}^{1/m} \quad \le \quad \left(\int_Q w_{i_0}^p \prod_{j=l+1}^m w_j^{p/p_j} \right)^{1/pm} \left(\int_Q \prod_{j=l+1}^m w_j^{-\frac{p}{p_j(pm-1)}} \right)^{1-1/pm}$$

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we obtain

$$\begin{aligned} \int_Q w_{i_0}^{1/m} &\leq \Big(\int_Q w_{i_0}^p \prod_{j=l+1}^m w_j^{p/p_j} \Big)^{1/pm} \Big(\int_Q \prod_{j=l+1}^m w_j^{-\frac{p}{p_j(pm-1)}} \Big)^{1-1/pm} \\ &\leq \Big(\int_Q w_{i_0}^p \prod_{j=l+1}^m w_j^{p/p_j} \Big)^{1/pm} \prod_{j=l+1}^m \Big(\int_Q w_j^{1-p_j'} \Big)^{\frac{1}{mp_j'}}. \end{aligned}$$

Proof: Case 1

This inequality combined with the $A_{\vec{P}}$ condition gives

$$\fint_Q w_{i_0}^{1/m} \leq \left(\fint_Q w_{i_0}^p \prod_{j=l+1}^m w_j^{p/p_j} \prod_{\substack{i=1\\ i\neq i_0}}^l w_i^p \prod_{\substack{i=1\\ i\neq i_0}}^l ess\, inf_Q\, w_i^{-p} \right)^{1/pm} \prod_{j=l+1}^m \left(\fint_Q w_j^{1-p_j'} \right)^{1/mp_j'}$$

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$$\begin{aligned}
\fint_Q w_{i_0}^{1/m} &\leq \left(\fint_Q w_{i_0}^p \prod_{j=l+1}^{m} w_j^{p/p_j} \prod_{\substack{i=1\\ i\neq i_0}}^{l} w_i^p \prod_{\substack{i=1\\ i\neq i_0}}^{l} ess\,inf_Q\, w_i^{-p} \right)^{1/pm} \prod_{j=l+1}^{m} \left(\fint_Q w_j^{1-p_j'} \right)^{1/mp_j'} \\
&\leq \left(\fint_Q \prod_{j=1}^{m} w_j^{p/p_j} \right)^{1/pm} \left(\prod_{j=1}^{l} ess\,inf_Q\, w_j^{-p} \prod_{j=l+1}^{m} \left(\fint_Q w_j^{1-p_j'} \right)^{p/p_j'} \right)^{1/pm} \\
&\times \left((ess\,inf_Q\, w_{i_0})^p \right)^{1/mp}
\end{aligned}$$

Proof: Case 1

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&\times ((ess\,inf_Q w_{i_0})^p)^{1/mp} \\
&\leq C w_{i_0}^{1/m}.
\end{aligned}
$$

Altogether proves that $\vec{w} \in A_{\vec{P}} \Rightarrow$ (B).

Proof: Case 1

$\boxed{(B) \Rightarrow A_{\vec{P}}}$

Now, set

$$\alpha = \frac{1}{1+pm(m-1)} \qquad \alpha_j = \frac{1/p+m(m-1)}{1/p_j+m-1}.$$

Using that $\sum_{j=1}^{m} \frac{1}{\alpha_j} = 1$, and Hölder's inequality, we obtain

$$\fint_Q v_{\vec{w}}^{\alpha} \le \prod_{j=1}^{m} \left(\fint_Q w_j^{\frac{\alpha p \alpha_j}{p_j}} \right)^{1/\alpha_j} = \prod_{j=1}^{m} \left(\fint_Q w_j^{\frac{1}{p_j(m-1)+1}} \right)^{\alpha p (m-1+1/p_j)} \tag{1}$$

Proof: Case 1

Using again Hölder's inequality, we get

$$1 = \fint_Q v_{\vec{w}}^{\frac{\alpha}{1+\alpha(pm-1)} - \frac{\alpha}{pm-1}\,\frac{pm-1}{1+\alpha(pm-1)}}$$

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Combining the above inequality and (1), we obtain for every weight w_j,

$$1 \leq \left(\fint_Q v_{\vec{w}}^{\alpha}\right)^{\frac{1}{\alpha p}} \left(\fint_Q v_{\vec{w}}^{\frac{-1}{pm-1}}\right)^{\frac{pm-1}{p}}$$

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1 &\leq \left(\fint_Q v_{\vec{w}}^{\alpha}\right)^{\frac{1}{\alpha p}} \left(\fint_Q v_{\vec{w}}^{\frac{-1}{pm-1}}\right)^{\frac{pm-1}{p}} \\
&\leq \prod_{j=1}^{m} \left(\fint_Q w_j^{\frac{1}{p_j(m-1)+1}}\right)^{m-1+1/p_j} \left(\fint_Q v_{\vec{w}}^{\frac{-1}{pm-1}}\right)^{m-1/p}
\end{aligned}
$$

Proof: Case 1

The previous inequality yields to the $A_{\vec{P}}$ condition. Indeed,

$$\left(\fint_Q \nu_{\vec{w}}\right)^{1/p} \prod_{i=1}^{m} \left(\fint_Q w_i^{1-p_i'}\right)^{1/p_i'}$$

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$$\leq [\nu_{\vec{w}}]_{A_{mp}}^{1/p} \prod_{i=1}^m [w_i^{1-p_i'}]_{A_{mp_i'}}^{1/p_i'}.$$

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\leq\quad &\left(\fint_Q \nu_{\vec{w}}\right)^{1/p} \left(\fint_Q \nu_{\vec{w}}^{\frac{-1}{mp-1}}\right)^{\frac{mp-1}{p}} \prod_{i=1}^m \left(\fint_Q w_i^{1-p_i'}\right)^{1/p_i'} \left(\fint_Q w_i^{\frac{-(1-p_i')}{mp_i'-1}}\right)^{m-1/p_i'} \\
\leq\quad &[\nu_{\vec{w}}]_{A_{mp}}^{1/p} \prod_{i=1}^m [w_i^{1-p_i'}]_{A_{mp_i'}}^{1/p_i'}.
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\leq\ & [v_{\vec{w}}]_{A_{mp}}^{1/p} \prod_{i=1}^m [w_i^{1-p_i'}]_{A_{mp_i'}}^{1/p_i'}.
\end{aligned}$$

Observe that,

$$[\vec{w}]_{A_P} \leq [v_{\vec{w}}]_{A_{mp}} \prod_{i=1}^m [\sigma_i]_{A_{mp_i'}}^{p/p_i'},$$

where $\sigma_i = w_i^{1-p_i'}$.

Proof: Case 2

Assume $p_j = 1$, for all $j = 1, \ldots, m$.

$A_{\vec{P}} \Rightarrow$ (B)

Assume $\vec{w} \in A_{(1,\ldots,1)}$, namely

$$\left(⨍_Q \left(\prod_{j=1}^m w_j \right)^{1/m} \right)^m \le c \prod_{j=1}^m ess\,inf_Q\, w_j,$$

and show that

$$\begin{cases} v_{\vec{w}} \in A_1 \\ w_j^{1/m} \in A_1,\ j = 1, \ldots, m. \end{cases}$$

Proof: Case 2

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$$\nu_{\vec{w}} \in A_1$$

It is clear since

$$\begin{aligned} \fint_Q \nu_{\vec{w}} = \fint_Q \prod_{i=1}^m w_i^{p/p_i} &= \fint_Q \prod_{i=1}^m w_i^{1/m} \\ &\leq C \prod_{i=1}^m ess\, inf_Q\, w_i^{1/m} \\ &\leq C \prod_{i=1}^m w_i^{1/m} = C \nu_{\vec{w}}. \end{aligned}$$

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\fint_Q \prod_{i=1}^m w_i^{1/m} \prod_{\substack{i=1 \\ i \neq j}}^m ess\, inf_Q\, w_i^{-1/m} &\leq C\, ess\, inf_Q\, w_j^{1/m} \\
\fint_Q w_j^{1/m} &\leq C w_j^{1/m}.
\end{aligned}$$

Proof: Case 2

$(B) \Rightarrow A_{\vec{P}}$

Combining the previous two conditions and Hölder's inequality, we get

$$\left(\fint_Q \left(\prod_{i=1}^{m} w_i \right)^{1/m} \right)^m \leq C \, ess\, inf_Q \prod_{i=1}^{m} w_i$$

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Proof: Case 2

$(\mathrm{B}) \Rightarrow A_{\vec{P}}$

Combining the previous two conditions and Hölder's inequality, we get

$$\begin{aligned}
\left(\fint_Q \left(\prod_{i=1}^m w_i \right)^{1/m} \right)^m &\leq C \, ess \, inf_Q \prod_{i=1}^m w_i \\
&\leq C \left(\fint_Q \left(\prod_{i=1}^m w_i \right)^{1/m^2} \right)^{m^2} \\
&\leq C \prod_{i=1}^m \left(\fint_Q w_i^{1/m} \right)^m
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which implies $\vec{w} \in A_{(1,\dots,1)}$. □

Remarks

Conditions in (B) are independent of each other.

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On one hand, set $\vec{w} = (w_1, w_1^{-p_2/p_1})$. Then

- $v_{\vec{w}} = 1$ which trivially belongs to A_{2p} for any w_1.
- Select w_1 so that $w_1^{-\frac{1}{p_1-1}} \notin L^1_{\text{loc}}$, then the first condition in (B) does not hold.

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Conversely, let $n = 1, m = 2$ and $p_1 = p_2 = 2$. Set $w_1 = w_2 = |x|^{-2}$.

- The first condition in (B) holds (because $w_j^{-1} = |x|^2 \in A_4$).
- $v_{\vec{w}} = |x|^{-2} \notin L^1_{\text{loc}}$, and hence $v_{\vec{w}} \notin A_2$.

Remarks

From a close inspection of the proof of the Proposition, it follows that

$$[\sigma_j]_{A_{mp'_j}} \lesssim [\vec{w}]_{A_{\vec{P}}}^{p'_j/p}$$

$$[\nu_{\vec{w}}]_{A_{mp}} \lesssim [\vec{w}]_{A_{\vec{P}}}$$

$$[\vec{w}]_{A_P} \lesssim [\nu_{\vec{w}}]_{A_{mp}} \prod_{i=1}^{m} [\sigma_i]_{A_{mp'_i}}^{p/p'_i}$$

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- $A_{(1,\ldots,1)}$ is contained in any $A_{\vec{P}}$ class.

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- No factorization or extrapolation theory for multiple weights.

Reference

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