Homework Set Topology

Juliette Kennedy

April 16, 2016

1 Homework Set 1

1. If $\{A_i\}$ and $\{B_i\}$ are two classes of sets such that $\{A_i\} \subseteq \{B_i\}$, then show that $\cup\{A_i\} \subseteq \cup\{B_i\}$ and $\cap\{B_i\} \subseteq \cap\{A_i\}$.
2. The *symmetric difference* of two sets A and B, denoted $A \triangle B$, is defined by $A \triangle B = (A - B) \cup (B - A)$. Show that
$(A \triangle B) \triangle C = A \triangle (B \triangle C)$
$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$
3. If f is a mapping from a set X into a set Y, show that
$f^{-1}(\emptyset) = \emptyset$; $f^{-1}(Y) = X$; $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$; $f^{-1}(\cup_i B_i = \cup_i f^{-1}(B_i)$; $f^{-1}(\cap_i B_i = \cap_i f^{-1}(B_i)$; $f^{-1}(B') = (f^{-1}(B))'$.
4. Consider the following relations: $m \leq n, m < n, m$ divides n. Are any of these equivalence relations? Are any of the relations transitive, or symmetric or reflexive?

2 Homework Set 2

1. The Cantor-Schröder-Bernstein Theorem states that if $A \preceq B$ and $B \preceq A$, then A and B are equinumerous. In the course of the proof given on page 29 of our textbook, the sets A and B are partitioned into sets A_i, A_e, A_o

and B_i, B_e, B_o, respectively. First, prove that these are partitions, i.e. prove that the disjoint union of A_i, A_e, A_o is A, and similarly for B. Describe the bijections that exist between the subsets of A and the subsets of B, and prove that they are bijections. How does this help you to finish the proof of the CSB Theorem?

2. Problem 2 on page 35: If $\{A_i\}_{i \in I}$ is a countable collection of countable sets (i.e. I is a countable set), prove that their union is countable, i.e. prove that $\cup_i A_i$ is countable.

3. Write 3/4 in binary and ternary notation.

4. Prove that the class of all subsets of the natural numbers is equinumerous with the half-open interval $[0, 1)$.

3 Homework Set 3

1. Let $X = \{a, b, c\}$ and let $d(x, y) = 0$, if $x = y$. $d(x, y) = 1$, otherwise. Prove that $\langle X, d \rangle$ is a metric space.

2. Problem 1, p. 69 of the text: Prove that any point and disjoint closed set in a metric space $\langle X, d \rangle$ can be separated by open sets, i.e. if x is a point and F is a closed set not containing x, then there exists a disjoint pair of open sets G_1, G_2 such that $x \in G_1$, $F \subseteq G_2$.

Prove that the statement obtained from the above by substituting a closed set for the point x is still true, i.e. any two closed sets of $\langle X, d \rangle$ can be separated by open sets.

3. Problem 2, p. 69 of the text: $\langle X, d \rangle$ is a metric space and $A \subseteq X$. If x is a limit point of A, show that each open sphere centred on x contains and infinite number of distinct points of A. Use this result to show that any finite subset of X is closed.

4. problem 7, p. 69 of the text: $\langle X, d \rangle$ is a metric space and $A \subseteq X$. Show that the complement of the closure of $A =$ the interior of the complement of

A. Show that the closure of $A = \{x | d(x, A) = 0\}$.

5. Describe the interior of the Cantor set. (Hint: if a point x is in the interior, then there is an open interval containing x which is entirely contained in the Cantor set. Hmm....can that happen?)

4 Homework Set 4

1. Prove that if f is a continuous function from the reals to the reals satisfying $f(x + y) = f(x) + f(y)$, the for some real number m, $f(x) = mx$.

2. Let X and Y be metric spaces and A a non-empty subset of X. If f and g are continuous mapping of X into Y such that $f(x) = g(x)$ for every x in A, show that $f(x) = g(x)$ for every x in $\bar{A}$.

3. Give an example of a function that is continuous but not uniformly continuous.

4. Prove that if X and Y are metric spaces and f is a mapping of X into Y, then f is continuous at x_0 if and only if $x_n \to x_0 \implies f(x_n) \to f(x_0)$.

5. Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

5 Homework Set 5

1. Let T_1 and T_2 be two topologies on a nonempty set X. Show that $T_1 \cap T_2$ is also a topology on X.

2. Let X be a nonempty set and consider the class of subsets of X which contain the empty set $\emptyset$ and all sets whose complements are countable. Is this a topology on X?

3. Let f be a continuous mapping of a topological space X into a topological space Y. If Z is a subspace of X, show that the restriction of f to Z is continuous.

4. Show that a subset of a topological space is dense if and only if it intersects every nonempty open set.
5. Show that a subset of a topological space is closed if and only if it contains its boundary.
6. Show that a subset of a topological space has empty boundary if and only if it is both open and closed.

6 Homework Set 6

1. State and prove Lindelöf's Theorem.
2. Let $f : X \to Y$ be a mapping of one topological space into another, and let B be an open base in X. Prove that f is continuous if and only if the inverse image of every basic open set is open.
3. Prove that any closed subspace of a compact space is compact.
4. Show that a continuous real function f defined on a compact space X attains its infimum and supremum, i.e. if $a = inf\{f(x) : x \in X\}$ and $b = sup\{f(x) : x \in X\}$ then there exists x_1 and x_2 in X such that $f(x_1) = a$ and $f(x_2) = b$.

7 Homework Set 7

1. In the Lebesgue Covering Lemma (p. 122 of the text), justify the assertion at the end of the proof that $B_{n_0} \subseteq S_r(x) \subseteq G_{i_0}$.
2. Show that a compact metric space is separable. HINT: Let $< X, d >$ be a compact metric space. For each n, consider the collection $\{S_{1/n}(x)\}_{x \in X}$. This is an open cover of $< X, d >$. By compactness, this open cover has a finite subcover. Now consider the set of all the centres of the spheres that make up this finite subcover. Try to find a countable dense subset of X from here.

3. Prove that the unit interval is compact using the finite intersection property characterisation of compactness: A topological space is compact if and only if every class of closed sets with the finite intersection property has non-empty intersection. (See p. 112 of the text.)

8 Homework Set 8

1. If $\{x_n\}$ is sequence of points in X, then the sequence is said to converge to a point x if for every neighbourhood G of x there exists a positive integer n_0 such that for all $n \geq n_0, x_n \in G$.

a.) Show that for the topological space with base set X and topology consisting of $\{\emptyset, X\}$, any sequence converges to every point of the space..

b.) Show that if X is a Hausdorff space then every convergent sequence has a unique limit.

2. Give an example of a topological space that is T_1 but not Hausdorff.

3. Prove that in a Hausdorff space any point and disjoint compact sets can be separated by open sets. (This is theorem C on p. 130.)

9 Homework Set 9

1. Problem 4 on p. 128: By considering the following sequence of functions, (a subclass of $C[0,1]$, i.e. all bounded continuous functions defined on the unit interval) defined by $f_n(x) = nx$ for $0 \leq x \leq 1/n$, $f_n(x) = 1$ for $1/n \leq x \leq 1$, show that $C[0,1]$ is not locally compact. (Definition: a space is *locally compact* if each of its points has a neighbourhood with compact closure.)

2. Show that any finite T_1 space is discrete.

3. If f is a continuous mapping of a topological space X into a Hausdorff space Y, show that the graph of f is a closed subset of the product space $X \times Y$