Homework Set 9, Topology 1, Solutions of exercises 1,2,3
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1. The space $C[0,1]$ of all bounded continuous functions defined on $[0,1]$ (equipped with the sup norm) is not locally compact.

Proof. Let $0(x) = 0$ for every $x \in [0,1]$ be the constant zero function. If we show that the closure of every neighbourhood of 0 is not compact then it follows that $C[0,1]$ is not locally compact.

Let $r > 0$ and suppose by contradiction that the closure of $S_r(0)$ is compact. Then it is also sequentially compact. Consider the sequence of functions $\{f_n\}$, where

$$f_n(x) = rnx \quad \text{if } 0 \leq x \leq \frac{1}{n},$$
$$f_n(x) = r \quad \text{if } \frac{1}{n} \leq x \leq 1.$$

Then f_n is continuous on $[0,1]$ and

$$||f_n|| = \sup_{x\in[0,1]} |f_n(x)| = r,$$

thus $\{f_n\}$ is a sequence in the closure of $S_r(0)$. Since this is sequentially compact, there exists a subsequence $\{f_{n_k}\}$ converging to some $f \in C[0,1]$. This means that for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$||f_{n_k} - f|| = \sup_{x\in[0,1]} |f_{n_k}(x) - f(x)| < \epsilon.$$

Thus f_{n_k} converges uniformly to f, which implies that it converges pointwise. Hence for every $x \in (0,1]$,

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) = r$$

since for every $x \in (0,1]$ there exists k_0 such that for all $k \geq k_0$, $\frac{1}{n_k} \leq x \leq 1$, which means $f_{n_k}(x) = r$ for every $k \geq k_0$.

Moreover,

$$f(0) = \lim_{k\to\infty} f_{n_k}(0) = 0.$$

Thus

$$\lim_{x\to 0^+} f(x) = r \neq 0 = f(0),$$

which means that f is not continuous at 0. This is a contradiction, hence the closure of $S_r(0)$ is not compact. □

2. Every finite T_1 space is discrete.

Proof. We want to show that every $A \subseteq X$ is open. If $A = \emptyset$ or $A = X$ then A is open so let us assume $A \neq X$, $A \neq \emptyset$.

Let $X = \{x_1, \ldots, x_n\}$ and let $A = \{x_{i_1}, \ldots, x_{i_k}\}$.

Then the complement of A is $X - \{x_{i_1}, \ldots, x_{i_k}\}$, which equals the set of the $n-k$ remaining points, call them $x_{j_1}, \ldots, x_{j_{n-k}}$. Thus $A' = \cup_{l=1}^{n-k}\{x_{j_l}\}$.

Since X is T_1, every $\{x_{j_l}\}$ is closed (by Thorem 26-A in the book), hence A' is closed (because it is a finite union of closed sets). This means that A is open. □

3. If X is a topological space, Y a Hausdorff space and $f : X \to Y$ is continuous, then the graph of f is a closed subset of the product space $X \times Y$.

Proof. Let $G = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ be the graph of f. We prove that its complement G' is open.

Let $(x, y) \in G'$, i.e. $y \neq f(x)$. Since Y is Hausdorff, there exist two disjoint open sets U and V in Y such that $y \in U$ and $f(x) \in V$.

Since f is continuous, $W = f^{-1}(V) \subseteq X$ is open.

Hence $W \times U$ is open in $X \times Y$ and $(x, y) \in W \times U$. It remains to show that $W \times U \subseteq G'$. If $(z, f(z)) \in G$ then if $z \notin W$ also $(z, f(z)) \notin W \times U$, if $z \in W$ then $f(z) \in V$ thus $f(z) \notin U$ (since V and U are disjoint). Hence $G \cap (W \times U) = \emptyset$, which means $W \times U \subseteq G'$.

We have shown that every point in G' has a neighbourhood which is contained in G', hence G' is open. □