Homework Set 8, Topology 1, Solutions of exercises 1,2
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1.

a) In the topological space with base set $X \neq \emptyset$ and topology $\mathbb{T} = \{\emptyset, X\}$, any sequence converges to every point of X.

Proof. Let $\{x_n\}$ be a sequence of points in X and let $x \in X$. Then the only neighbourhood G of x is X (since $x \notin \emptyset$), thus for every n, $x_n \in G = X$, which means that $\{x_n\}$ converges to x. □

b) If $(X, \mathbb{T})$ is Hausdorff then every convergent sequence has a unique limit.

Proof. Let $\{x_n\}$ be a convergent sequence in X. Suppose that it has two limits $x_1 \neq x_2$. Since X is Hausdorff, there exist $G_1, G_2 \in \mathbb{T}$ such that $x_1 \in G_1$, $x_2 \in G_2$ and $G_1 \cap G_2 = \emptyset$.

By definition of convergence, there exists n_1 such that $x_n \in G_1$ for every $n \geq n_1$ and there exists n_2 such that $x_n \in G_2$ for every $n \geq n_2$. Hence for every $n \geq \max\{n_1, n_2\}$, $x_n \in G_1 \cap G_2$, which is a contradiction since $G_1 \cap G_2 = \emptyset$. Thus $x_1 = x_2$, that is the limit is unique. □

2. An example of a topological space that is T_1 but not Hausdorff is the following. Let X be any infinite set and let $\mathbb{T} = \emptyset \cup \{A \subseteq X : A' \text{ is finite}\}$.

To show that X is T_1, let $x, y \in X$ and let $G = \{z \in X : z \neq x\} = \{x\}'$ and $H = \{z \in X : z \neq y\} = \{y\}'$. Then $G, H \in \mathbb{T}$ since their complements are $\{x\}$ and $\{y\}$, which are finite. Moreover, $y \in G$ but $x \notin G$ and $x \in H$ but $y \notin H$.

To see that X is not Hausdorff, we prove that for any $x, y \in X$ any two neighbourhoods $G_1, G_2 \in \mathbb{T}$, $x \in G_1$, $y \in G_2$, cannot be disjoint. Indeed, suppose $G_1 \cap G_2 = \emptyset$. Then $G_1' \cup G_2' = (G_1 \cap G_2)' = X$, but this is a contradiction since G_1' and G_2' are finite (and so is their union) whereas X is infinite.