Homework Set 7, Topology 1, Solutions of exercises 1,2
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1. To conclude the Lebesgue Covering Lemma it suffices to show the following.

Lemma 0.1. *Let (X, d) be a metric space, let $x \in X$ and $r > 0$. Let $A \subseteq X$ be such that $d(A) < r$ and $A \cap S_r(x) \neq \emptyset$. Then $A \subseteq S_{2r}(x)$.*

Proof. Let $y \in A$. Then for every $z \in A \cap S_r(x)$, we have by triangle inequality

$$d(y, x) \leq d(y, z) + d(z, x).$$

Since $z, y \in A$, $d(y, z) \leq d(A) < r$. Moreover, $z \in S_r(x)$, thus $d(z, x) < r$. It follows that $d(y, x) < 2r$, hence $y \in S_{2r}(x)$. □

In the end of the proof of the Lebesgue Covering Lemma (p. 122), we have B_{n_0} such that $d(B_{n_0}) < r/2$ and $x_{n_0} \in B_{n_0} \cap S_{r/2}(x)$. Applying the lemma with $A = B_{n_0}$, we conclude that $B_{n_0} \subseteq S_r(x)$.

2. Any compact metric space (X, d) is separable.

Proof. For each natural number n, the collection $\{S_{1/n}(x)\}_{x \in X}$ is an open cover of X. By compactness, there exists a finite subcover, call it $\{S_{1/n}(x_1^n), \ldots, S_{1/n}(x_{k_n}^n)\}$. Let $A_n = \{x_1^n, \ldots, x_{k_n}^n\}$. Then $A = \cup_n A_n$ is countable and we want to show that it is dense in X.

For this purpose it suffices to show that $X \subseteq \bar{A}$ (since $\bar{A} \subseteq X$ always). Let $y \in X$. For every $\epsilon > 0$, there exists n such that $1/n < \epsilon$. Since $\{S_{1/n}(x_1^n), \ldots, S_{1/n}(x_{k_n}^n)\}$ covers X, there exists x_i^n such that $y \in S_{1/n}(x_i^n)$, thus $d(x_i^n, y) < 1/n < \epsilon$. Since $x_i^n \in A$, it follows that $S_\epsilon(y) \cap A \neq \emptyset$ for every ϵ. Hence $y \in \bar{A}$.

Thus X has a countable dense subset, which means that X is separable. □

3. Prove that the unit interval is compact using the finite intersection property characterisation of compactness.

Proof. See Theorem 21-G (p. 114) in the textbook (and Theorems 21-D,E,F for preliminaries). □