Homework Set 6, Topology 1, Solutions of exercises 2,3,4
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2. Let (X, T_1), (Y, T_2) be topological spaces, let $f : X \to Y$ be a mapping and let $\mathcal{B}$ be an open base in Y. Then f is continuous if and only if $f^{-1}(B) \in T_1$ for every $B \in \mathcal{B}$.

Proof. If f is continuous then by definition for every $U \in T_2$, $f^{-1}(U) \in T_1$. Since every $B \in \mathcal{B}$ belongs to T_2, it follows that $f^{-1}(B) \in T_1$.

On the other hand, suppose that f is such that $f^{-1}(B) \in T_1$ for every $B \in \mathcal{B}$. Let $U \in T_2$. Since $\mathcal{B}$ is an open base, U can be written as $U = \cup_{i \in I} B_i$ for some $B_i \in \mathcal{B}$. But then $f^{-1}(U) = \cup_{i \in I} f^{-1}(B_i)$ and by assumption $f^{-1}(B_i) \in T_1$ for every $i \in I$. Since T_1 is a topology, this implies that $\cup_{i \in I} f^{-1}(B_i) \in T_1$, thus $U \in T_1$. This means that f is continuous. □

3. Let (X, T) be a compact topological space and let $F \subseteq X$ be a closed subspace. Then F is compact.

Proof. Let $\cup_{i \in I} U_i$ be an open cover of F. Since U_i are open in the subspace topology, it follows that $U_i = V_i \cap F$ for some $V_i \in T$ for every $i \in I$. Then $F' \cup (\cup_{i \in I} U_i)$ is an open cover of X because F' is open (F is closed) and $\cup_{i \in I} U_i$ covers F. Since X is compact, this cover has a finite subcover $F' \cup (\cup_{n=1}^{N} V_n)$ (it can also happen that F' does not belong to this subcover). Then $\cup_{n=1}^{N} (V_n \cap F)$ is a finite subcover of F, hence F is compact. □

4. Let (X, T) be a compact topological space and let $f : X \to \mathbb{R}$ be continuous. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = \inf\{f(x) : x \in X\}$ and $f(x_2) = \sup\{f(x) : x \in X\}$.

Proof. Since X is compact and f is continuous, $f(X)$ is compact by Theorem $21 - B$ in Simmon's book. Hence $f(X) \subset \mathbb{R}$ is closed and bounded by the inverse of the Heine-Borel theorem. It follows that f has infimum and supremum, let $a = \inf\{f(x) : x \in X\}$ and $b = \sup\{f(x) : x \in X\}$.

Let (c, d) be any open interval containing a. Since $a < d$, by definition of infimum there exists $x \in X$ such that $a \leq f(x) < d$. Thus $(c, d) \cap f(X) \neq \emptyset$, which implies (by definition of closure of a set) that a is in the closure of $f(X)$, which equals $f(X)$ since $f(X)$ is closed. Hence $a = f(x_1)$ for some $x_1 \in X$.

Similarly one can can show that $b = f(x_2)$ for some $x_2 \in X$. □