Homework Set 5, Topology 1, Solutions of exercises 2,3,4,5,6
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2. Let $X \neq \emptyset$. Then $\mathbb{T} = \emptyset \cup \{A \subset X : A' \text{ is countable}\}$ is a topology on X.

Proof. We prove that $\mathbb{T}$ is closed under the formation of arbitrary unions and finite intersections.

(1) Let $A_i \in \mathbb{T}$ for $i \in \mathcal{I}$ (any index set). If $A_i = \emptyset$ for every i then $\cup_{i\in\mathcal{I}} A_i = \emptyset \in \mathbb{T}$. Otherwise, there exists at least one k such that A'_k is countable. Since

$$(\cup_{i\in\mathcal{I}} A_i)' = \cap_{i\in\mathcal{I}} A'_i \subseteq A'_k,$$

it follows that $(\cup_{i\in\mathcal{I}} A_i)'$ is countable, that is $\cup_{i\in\mathcal{I}} A_i \in \mathbb{T}$.

(2) Let $A_n \in \mathbb{T}$, $n = 1, \ldots, N$. If $A_n \neq \emptyset$ for every n then A'_n is countable for every n. Since

$$(\cap_{n=1}^N A_n)' = \cup_{n=1}^N A'_n$$

and the finite union of countable sets is countable, we have that $(\cap_{n=1}^N A_n)'$ is countable, hence $\cap_{n=1}^N A_n \in \mathbb{T}$. If $A_n = \emptyset$ for some n then $\cap_{n=1}^N A_n = \emptyset \in \mathbb{T}$.

□

3. Let $(X, \mathbb{T}_1)$ and $(Y, \mathbb{T}_2)$ be topological spaces and let $f : X \to Y$ be continuous. If Z is a subspace of X, then the restriction of f to Z is continuous.

Proof. Let $A \subset Y$ be open, that is $A \in \mathbb{T}_2$. Since f is continuous, $f^{-1}(A)$ is open in X, that is $f^{-1}(A) \in \mathbb{T}_1$.

But Z is a subspace of X, hence $f^{-1}(A) \cap Z$ is open in Z (by the definition of subspace topology). This means that the restriction of f to Z is continuous. □

4. Let $(X, \mathbb{T})$ be a topological space and let $A \subseteq X$. Then A is dense if and only if $A \cap B \neq \emptyset$ for every $B \in \mathbb{T}$, $B \neq \emptyset$.

Proof. First assume that A is dense, that is $\bar{A} = X$. Let $B \in \mathbb{T}$, $B \neq \emptyset$. Suppose by contradiction that $A \cap B = \emptyset$, thus $A \subseteq B'$. Since $\bar{A}$ is the intersection of all closed supersets of A, we have $B' = X$. But this means $B = \emptyset$, which is a contradiction. Thus $A \cap B \neq \emptyset$.

Assume now that for every $B \in \mathbb{T}$, $B \neq \emptyset$, we have $A \cap B \neq \emptyset$. Let C be a closed superset of A, that is $C' \in \mathbb{T}$, $A \subseteq C$. Suppose $C' \neq \emptyset$. Then by assumption $A \cap C' \neq \emptyset$, which is a contradiction since $A \subseteq C$. Thus $C = X$, which implies $\bar{A} = X$. □

5. Let $(X, \mathbb{T})$ be a topological space and let $A \subseteq X$. Then A is closed if and only if it contains its boundary.

Proof. The boundary of A is by definition $\bar{A} \cap \bar{A}'$.

If A is closed, then $A = \bar{A}$ thus the boundary of A is $A \cap \bar{A}' \subseteq A$.

Assume now that the boundary of A is contained in A. Suppose by contradiction that $\bar{A} - A \neq \emptyset$ and let $x \in \bar{A} - A$. Then x is a limit point of A, thus given any neighbourhood N of x we have $N \cap A \neq \emptyset$. But $N \cap A' \neq \emptyset$ since it contains x, hence x is in the boundary of A. By assumption this implies $x \in A$, which is a contradiction. Thus $\bar{A} = A$, which means that A is closed. □

6. Let $(X, \mathbb{T})$ be a topological space and let $A \subseteq X$. Then A has empty boundary if and only if it is both open and closed.

Proof. Let A be open and closed. Since A is closed then $\bar{A} = A$. Since A is open, A' is closed thus $\bar{A}' = A'$. But the boundary of A equals $\bar{A} \cap \bar{A}' = A \cap A' = \emptyset$.

Assume now that the boundary of A is empty. Then it is contained in A, thus by ex.5 A is closed. On the other hand, the boundary of A' is also empty since it equals the boundary of A. Hence the boundary of A' is contained in A', which by ex.5 is thus closed. But this implies that A is open. □