Homework Set 4, Topology 1, Solutions of exercises 1,2,5
Laura Venieri

1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x+y) = f(x) + f(y)$ then $f(x) = mx$ for some $m \in \mathbb{R}$.

Proof. Let $g(x) = f(x) - f(1)x$. If we show that $g(x) = 0$ for every $x \in \mathbb{R}$ then $f(x) = mx$ with $m = f(1)$.

First observe that since $f(x+y) = f(x) + f(y)$ we have $g(x+y) = g(x) + g(y)$.

Moreover, g is continuous. Indeed, let $x \in \mathbb{R}$ and let $\epsilon > 0$. Then (by the continuity of f) there exists $\delta_1 > 0$ such that $|x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \epsilon/2$. Choose $\delta = \min\{\delta_1, \epsilon/(2|f(1)|)\}$. Then for $|x-y| < \delta$ we have

$$|g(x) - g(y)| \leq |f(x) - f(y)| + |f(1)||x-y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

First we show by induction that $g(n) = 0$ for every $n \in \mathbb{N}$. Indeed, $g(0) = g(0+0) = g(0) + g(0)$ thus $g(0) = 0$; $g(1) = f(1) - f(1) = 0$. Suppose now that $g(n-1) = 0$. Then $g(n) = g(n-1+1) = g(n-1) + g(1) = 0$.

Since $0 = g(0) = g(n-n) = g(n) + g(-n)$, we have $g(-n) = -g(n) = 0$ for every $n \in \mathbb{N}$. Thus $g(n) = 0$ for every $n \in \mathbb{Z}$.

Next we show that $g(r) = 0$ for every $r \in \mathbb{Q}$. Let $r = \frac{p}{q}$ with $p, q \in \mathbb{Z}$. Then

$$0 = g(p) = g\left(q\frac{p}{q}\right) = qg\left(\frac{p}{q}\right),$$

thus $g(r) = 0$.

Now we use the density of $\mathbb{Q}$ in $\mathbb{R}$ and the continuity of g to show that $g(x) = 0$ for every $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that $|x-y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $|x-r| < \delta$. Thus $|g(x) - g(r)| = |g(x)| < \epsilon$, which implies $g(x) = 0$. □

2. Let (X, d_1) and (Y, d_2) be metric spaces and let $A \subseteq X$, $A \neq \emptyset$. If $f, g : X \to Y$ are continuous and $f(x) = g(x)$ for every $x \in A$ then $f(x) = g(x)$ for every $x \in \bar{A}$.

Proof. Let $x \in \bar{A}$. Then either $x \in A$ or x is a limit point of A.

If $x \in A$ then $f(x) = g(x)$ by assumption. Let now x be a limit point of A. Let $\epsilon > 0$. Then by continuity of f there exists $\delta_1 > 0$ such that $d_1(x,y) < \delta_1 \Rightarrow d_2(f(x), f(y)) < \epsilon/2$, and by continuity of g there exists $\delta_2 > 0$ such that $d_1(x,y) < \delta_2 \Rightarrow d_2(g(x), g(y)) < \epsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Since x is a limit point of A, there exists $y \in A$ such that $d_1(x,y) < \delta$. Thus we have $f(y) = g(y)$ and by triangle inequality

$$d_2(f(x), g(x)) \leq d_2(f(x), f(y)) + d_2(g(y), g(x)) < \epsilon.$$

Since this holds for arbitrary ϵ, we have $d_2(f(x), g(x)) = 0$ thus $f(x) = g(x)$. □

5. A Cauchy sequence in a metric space (X, d) is convergent if and only if it has a convergent subsequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence in X.

If $\{x_n\}$ is convergent then it is a convergent subsequence of itself.

Assume now that $\{x_n\}$ has a convergent subsequence, call this $\{x_{n_k}\}$. Let $x_{n_k} \to x_0$. We will show that also $x_n \to x_0$.

Let $\epsilon > 0$. Then there exists n_0 such that $d(x_{n_k}, x_0) < \epsilon/2$ for all $n_k \geq n_0$. Moreover, since $\{x_n\}$ is a Cauchy sequence, there exists N_0 such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N_0$. Let now $N = \max\{n_0, N_0\}$ and let $n_k > N$. Then for every $n \geq N$,

$$d(x_n, x_0) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $x_n \to x_0$. $\square$