Homework Set 3, Topology 1, Solutions of exercises 2,3,4,5 (and one extra)

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2. If (X, d) is a metric space, $x \in X$, $F \subseteq X$ is a closed set with $x \notin F$, then there exist two disjoint open sets G_1, G_2 such that $x \in G_1$ and $F \subseteq G_2$.

Moreover, if F_1, F_2 are two disjoint closed sets then there exist two disjoint open sets G_1, G_2 such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.

Proof. Since for every $x \in X$, $\{x\}$ equals the closed ball of radius 0 and center x, we have that $\{x\}$ is closed.

Thus the first claim is a special case of the second one and it suffices to prove the second one.

First observe that if F is closed and $x \notin F$ then $d(x, F) > 0$. Indeed, suppose by contradiction that $d(x, F) = 0$. Since $d(x, F) = \inf\{d(x, y) : y \in F\}$, for every $r > 0$ there exists $y \in F$ such that $d(x, y) < r$, that is $y \in S_r(x)$. Since $x \notin F$, y is distinct from x thus x is a limit point of F. But this is a contradiction because F is closed so it contains all its limit points.

For every $x \in F_1$, let $r_x = \frac{d(x,F_2)}{2}$ and for every $y \in F_2$, let $r_y = \frac{d(y,F_1)}{2}$. Let

$$G_1 = \bigcup_{x \in F_1} S_{r_x}(x), \qquad G_2 = \bigcup_{y \in F_2} S_{r_y}(y).$$

Then G_1 and G_2 are open sets because they are unions of open balls. Moreover, $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.

It remains to show that $G_1 \cap G_2 = \emptyset$. Suppose by contradiction that there exists a point $z \in G_1 \cap G_2$. Then there exist $x \in F_1$ and $y \in F_2$ such that $z \in S_{r_x}(x) \cap S_{r_y}(y)$. By triangle inequality, we have

$$d(x, y) \leq d(x, z) + d(z, y) < r_x + r_y. \tag{1}$$

But $d(x, y) \geq d(x, F_2) = 2r_x$ and $d(x, y) \geq d(F_1, y) = 2r_y$ and summing these two and dividing by 2 we get

$$d(x, y) \geq r_x + r_y,$$

which yields a contradiction with (1). □

3. Let (X, d) be a metric space and $A \subseteq X$. Then

(a) If x is a limit point of A, then for every $r > 0$, $S_r(x)$ contains infinitely many points of A.

(b) Any finite subset of X is closed.

Proof. (a) Let $r > 0$ and suppose by contradiction that $(S_r(x) \cap A) - \{x\} = \{a_1, \dots, a_n\}$ (it is $\neq \emptyset$ since x is a limit point of A). Let $R = \min\{d(a_i, x) : i = 1, \dots, n\}$. Then $d(x, a_i) \geq R$ for every $i = 1, \dots, n$, thus $S_R(x)$ contains no point of A different from x. This yields a contradiction since x is a limit point of A.

(b) Let A be finite. By (a), if x is a limit point of A then any open sphere $S_r(x)$ contains infinitely many points of A. But A contains only finitely many points, thus A has no limit points. It follows that A contains all its limit points, hence it is closed by definition.

□

4. Let (X, d) be a metric space and $A \subseteq X$. Then

(a) $(\bar{A})' = \mathrm{Int}(A')$;

(b) $\bar{A} = \{x \in X : d(x, A) = 0\}$.

Proof. (a) Since $\bar{A}$ is closed, $(\bar{A})'$ is open. Moreover, $A \subseteq \bar{A}$ implies $(\bar{A})' \subseteq A'$. Thus $(\bar{A})' \subseteq \mathrm{Int}(A')$ (because $\mathrm{Int}(A')$ contains every open subset of A').

On the other hand, if $x \in \mathrm{Int}(A')$ then there exists $r > 0$ such that $S_r(x) \subset A'$. Thus $S_r(x) \cap A = \emptyset$, that is x is not a limit point of A. This implies that $x \notin \bar{A}$, hence $x \in (\bar{A})'$. Thus also $\mathrm{Int}(A') \subseteq (\bar{A})'$.

(b) By definition, $x \in \bar{A}$ if and only if $x \in A$ or x is a limit point of A. If $x \in A$ then $d(x, A) = 0$. The fact that x is a limit point of A is equivalent to say that for every $r > 0$ there exists $y \neq x$ such that $y \in A \cap S_r(x)$. This happens if and only if for every $r > 0$ there exists $y \in A$ such that $d(x, y) < r$, which is equivalent to say that $d(x, A) < r$ for every $r > 0$. But this means that $d(x, A) = 0$.

□

5. The interior of the Cantor set F is empty.

Proof. By definition, $F = \cap_{n=1}^{\infty} F_n$, where $F_1 = [0, 1]$, $F_2 = [0, 1/3] \cup [2/3, 1]$, ... Each F_n is the union of 2^{n-1} closed intervals of length 3^{1-n} each.

Suppose by contradiction that there exists $x \in \mathrm{Int}(F)$. Then there exists $r > 0$ such that $S_r(x) = (x - r, x + r) \subseteq F$. Thus $S_r(x) \subseteq F_n$ for every n, that is $S_r(x)$ is contained in one of the intervals whose union is F_n for every n. Hence $0 < 2r \leq 3^{1-n}$ for every n, which yields a contradiction since the right hand side tends to 0 as $n \to \infty$. □

Extra exercise: A subset A of a metric space X is nowhere dense if and only if every non-empty open subset of X contains an open sphere disjoint from A.

Proof. Suppose that A is nowhere dense. Let $U \subseteq X$ be open, $U \neq \emptyset$. Since $\text{Int}(\bar{A}) = \emptyset$, U is not contained in $\text{Int}(\bar{A})$, thus U is not contained in $\bar{A}$. This means that there exists $b \in U$ such that $b \notin \bar{A}$, hence $b \in U \cap (\bar{A})'$. Call this set W. Since W is open, there exists an open sphere $S_r(b) \subseteq W$. Thus $S_r(b) \cap A = \emptyset$ and the claim is proved.

The other direction is straightforward and was seen in Juliette's lecture. □