Homework Set 2, Topology 1, Solutions of exercises 1,2,3
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1. Cantor-Schröder-Bernstein Theorem: if $A \preceq B$ and $B \preceq A$ then A and B are equinumerous.

Proof. (Completing the details of the proof on page 29 of the textbook)

By assumption, there exist a one-to-one mapping (i.e. injective) $f : A \to B$ and a one-to-one mapping $g : B \to A$. We want to find a one-to-one correspondence $F : A \to B$, that is a bijection.

Given $x \in A$, we define its ancestors as in the textbook and also A_i, A_e and A_o. We want to show that they form a partition of A. First observe that they are disjoint by definition: if $x \in A_i$ then it has an infinite number of ancestors so $x \notin A_e \cup A_o$; if $x \in A_e$ then it has an even number of ancestors (so in particular finite), hence $x \in A_i \cup A_o$ and so on.

Observe that since f g are injective, so are f^{-1} and g^{-1}. Hence for every $x \in A$ the ancestors are uniquely determined.

It remains to show that $A = A_i \cup A_e \cup A_o$.

We prove the double inclusion. First let $x \in A$. If x has zero ancestors then $x \in A_e$. Otherwise, x has at least one ancestor. It can happen either that x has finitely many ancestors n or infinitely many. In the first case, $x \in A_e$ or A_o depending on whether n is even or odd. In the second case, $x \in A_i$. Thus $A \subseteq A_i \cup A_e \cup A_o$.

The other inclusion $A \subseteq A_i \cup A_e \cup A_o$ is obvious by definition of the sets A_i, A_e, A_o.

Similarly, the sets B_i, B_e and B_o partition B.

There are now three bijections between these subsets:

i) f is a bijection from A_i to B_i. Indeed, by assumption f is injective so we need to prove that it is also surjective. Let $y \in B_i$. Since y has infinitely many ancestors, there is $x \in A$ such that $f^{-1}(y) = x$. But since y has infinitely many ancestors, also x will have infinitely many ancestors so $x \in A_i$. Thus $y = f(x)$ with $x \in A_i$ and f is surjective.

ii) f is a bijection from A_e to B_o. Again we need to prove that the mapping is surjective. Let $y \in B_o$, thus y has at least one ancestor $f^{-1}(y) = x \in A$. If y has $2n + 1$ ancestors then x will have $2n$ ancestors, thus $x \in A_e$. Hence $y = f(x)$ with $x \in A_e$.

iii) g^{-1} is a bijection from A_o to B_e. Let $y \in B_e$. We want to show that $g(y) \in A_o$, so $y = g^{-1}(g(y))$ and the mapping is surjective. Suppose by contradiction that $g(y) \in A_i \cup A_e$. If $g(y) \in A_i$ then it has infinitely many ancestors, so also $y = g^{-1}(g(y))$ has infinitely many ancestors. Hence $y \in B_i$, which is a contradiction. If $g(y) \in A_e$ then $g(y)$ has $2n$ ancestors ($n \geq 1$), so $y = g^{-1}(g(y))$ has $2n - 1$ ancestors. This implies $y \in B_o$, which is a contradiction.

Now we are ready to construct F as

$$F(x) = \begin{cases} f(x) & \text{if } x \in A_i \cup A_e, \\ g^{-1}(x) & \text{if } x \in A_o. \end{cases}$$

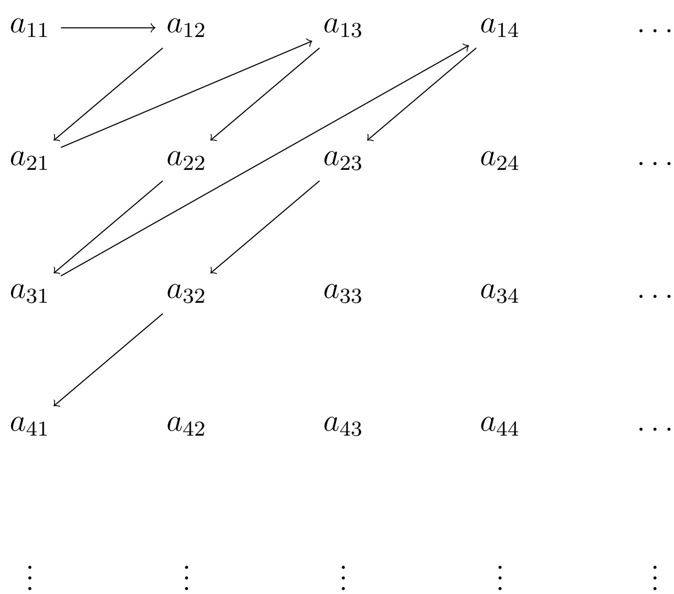
Then $F : A \to B$ is a bijection because A_i, A_e and A_o partition A, B_i, B_e and B_o partition B and by i), ii) and iii) $F \restriction_{A_i}: A_i \to B_i$, $F \restriction_{A_e}: A_e \to B_o$ and $F \restriction_{A_o}: A_o \to B_e$ are bijections. $\square$

2. If $\{A_i\}_{i \in I}$ is a countable collection of countable sets, then $\cup_{i\in I} A_i$ is countable.

Proof. We use the idea behind Figure 13 in Simmon's book. For each $i \in I$, we can enumerate the elements in A_i (since there are countably many of them) as

$$A_i = \{a_{ij} : j \in \mathbb{N}\}.$$

We can then enumerate the elements in $\cup_{i\in I} A_i$ as in the following picture:



Thus a_{11} will correspond to number 1, a_{12} to number 2, a_{21} to number 3, a_{13} to number 4 and so on following the arrows in the picture. $\square$

3. To write $\frac{3}{4}$ in binary notation, first split the interval $[0, 1)$ into two subintervals of length $1/2$: $[0, 1) = [0, 1/2) \cup [1/2, 1)$ and assign number 0 to the first interval and number 1 to the second (these will correspond to the digits in the binary expansion). Since $\frac{3}{4} \in [1/2, 1)$ then the first digit in the expansion will be 1.

Then split again $[1/2, 1) = [1/2, 3/4) \cup [3/4, 1)$ and since $\frac{3}{4}$ belongs to the second interval, the second digit in the expansion is 1.

Continuing by splitting $[3/4, 1)$ into two subintervals (and so on) we will always have that $\frac{3}{4}$ belongs to first interval, so it is $0.110000\ldots$ in binary notation.

Indeed, we can verify $\frac{3}{4} = 1\frac{1}{2} + 1\frac{1}{2^2}$.

To write $\frac{3}{4}$ in ternary notation, we use the same method dividing at each step the interval to which $\frac{3}{4}$ belongs into 3 subintervals of the same length and assign them the numbers $0, 1, 2$ in order.

Thus $[0, 1) = [0, 1/3) \cup [1/3, 2/3) \cup [2/3, 1)$ and since $\frac{3}{4}$ belongs to the third interval its first digit will be 2.

Then $[2/3, 1) = [2/3, 7/9) \cup [7/9, 8/9) \cup [8/9, 1)$. Since $\frac{3}{4}$ belongs to the first interval, its second digit is 0.

Again we split $[2/3, 7/9) = [2/3, 19/27) \cup [19/27, 20/27) \cup [20/27, 7/9)$. Then $\frac{3}{4}$ belongs to the third interval so the third digit is 2.

Actually one can verify that $\frac{3}{4}$ in ternary notation is $0.2020202020\ldots$ either by induction using the above method or just because

$$\frac{3}{4} = \sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^{2n+1}$$

by the sum of a geometric series with odd powers.

4. The class of all subsets of the natural numbers $\mathcal{P}(\mathbb{N})$ is equinumerous with $[0, 1)$.

I will not write the details here but we can construct two injective functions $f : \mathcal{P}(\mathbb{N}) \to [0, 1)$ and $g : [0, 1) \to \mathcal{P}(\mathbb{N})$ for example as it is done in Simmon's book on pages 41-42 and then conclude by using the Cantor-Schröder-Bernstein theorem.