Homework Set 1, Topology 1, Solutions to exercises 2 and 3

2. First we prove that

$$(A \triangle B) \triangle C = A \triangle (B \triangle C). \tag{1}$$

Proof. By definition,

$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) &(2)\\ &= (A \cap B') \cup (A' \cap B) &(3)\\ &= (A \cup B) \cap (A' \cup B'). &(4) \end{aligned}$$

Here (4) follows from (3) using the distributive laws, because

$$\begin{aligned} &(A \cap B') \cup (A' \cap B)\\ &= ((A \cap B') \cup A') \cap ((A \cap B') \cup B)\\ &= ((A \cup A') \cap (B' \cup A')) \cap ((A \cup B) \cap (B' \cup B))\\ &= (B' \cup A') \cap (A \cup B). \end{aligned}$$

The left-hand side in (1) equals by (3)

$$(((A \cap B') \cup (A' \cap B)) \cap C') \cup ((A \triangle B)' \cap C),$$

which using (4) becomes

$$(((A \cap B') \cup (A' \cap B)) \cap C') \cup (((A \cup B) \cap (A' \cup B'))' \cap C).$$

Using now the distributive laws and properties of complement, we get

$$(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (((A' \cap B') \cup (A \cap B)) \cap C),$$

which using again the distributive laws equals to

$$(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A \cap B \cap C). \tag{5}$$

For the right-side of (1), using the same tools we get

$$\begin{aligned} &(A \cap (B \triangle C)') \cup (A' \cap ((B \cap C') \cup (B' \cap C)))\\ &= (A \cap ((B \cup C) \cap (B' \cup C'))') \cup (A' \cap ((B \cap C') \cup (B' \cap C)))\\ &= (A \cap ((B \cup C)' \cup (B' \cup C')')) \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)\\ &= (A \cap ((B' \cap C') \cup (B \cap C))) \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)\\ &= (A \cap B' \cap C') \cup (A \cap B \cap C) \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C), \end{aligned}$$

which equals (5) since the union is commutative. □

We now prove that $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

Proof. We start from the right-hand side, use the definition of symmetric difference, the distributive laws of intersection over minus and union to get

$$\begin{aligned}
&(A \cap B) \triangle (A \cap C) \\
&= ((A \cap B) - (A \cap C)) \cup ((A \cap C) - (A \cap B)) \\
&= ((A \cap (B - C)) \cup (A \cap (C - B)) \\
&= A \cap ((B - C) \cup (C - B)) \\
&= A \cap (B \triangle C).
\end{aligned}$$

The distributive law of intersection over minus (used to go from the second line to the third) is the following

$$(A - B) \cap C = (A \cap C) - (B \cap C).$$

It holds because using De Morgan's law $(A - (B \cap C) = (A - B) \cup (B - C))$ we have

$$\begin{aligned}
(A \cap C) - (B \cap C) &= ((A \cap C) - B) \cup ((A \cap C) - C) \\
&= ((A \cap C) - B) \cup \emptyset \\
&= (A \cap C) \cap B' \\
&= (A \cap B') \cap C \\
&= (A - B) \cap C.
\end{aligned}$$

□

3. If $f : X \to Y$ then

- $f^{-1}(\emptyset) = \emptyset$ because by definition $f^{-1}(\emptyset) = \{x \in X : f(x) \in \emptyset\} = \emptyset$ since there is no such x.
- $f^{-1}(Y) = X$ because for every $x \in X$, $f(x) \in Y$.
- $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$ because if $y \in f^{-1}(B_1)$ then $y = f(x)$ with $x \in B_1$, thus $x \in B_2$, which implies $y \in f^{-1}(B_2)$.
- $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$

  We prove the double inclusion. If $y \in f^{-1}(\cup_i B_i)$ then $y = f(x)$ with $x \in \cup_i B_i$. Thus $x \in B_i$ for at least one i, which implies that $y \in f^{-1}(B_i)$ for at least one i. Hence $y \in \cup_i f^{-1}(B_i)$.

  On the other hand, if $y \in \cup_i f^{-1}(B_i)$, then $y \in f^{-1}(B_i)$ for at least one i. Thus $y = f(x)$ with $x \in B_i$ for at least one i, that is $x \in \cup_i B_i$. Hence $y \in f^{-1}(\cup_i B_i)$.

- $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$

  Again with double inclusion. If $y \in f^{-1}(\cap_i B_i)$ then $y = f(x)$ with $x \in \cap_i B_i$, that is $x \in B_i$ for every i. Hence $y \in f^{-1}(B_i)$ for every i, which means $y \in \cap_i f^{-1}(B_i)$.

  On the other hand, if $y \in \cap_i f^{-1}(B_i)$ then $y \in\in f^{-1}(B_i)$ for every i, thus $y = f(x)$ with $x \in B_i$ for every i. This implies $x \in \cap_i B_i$, hence $y \in f^{-1}(\cap_i B_i)$.

- $f^{-1}(B') = (f^{-1}(B))'$

  If $y \in f^{-1}(B')$ then $y = f(x)$ with $x \in B'$, that is $x \notin B$. Thus $y \notin f^{-1}(B)$, which means $y \in (f^{-1}(B))'$. Hence $f^{-1}(B') \subseteq (f^{-1}(B))'$.

  If $y \in (f^{-1}(B))'$ then $y \notin f^{-1}(B)$, that is $y = f(x)$ with $x \notin B$. This implies $x \in B'$, thus $y \in f^{-1}(B')$. Hence $(f^{-1}(B))' \subseteq f^{-1}(B')$.