SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 9

Exercise 1. Consider maps $f = (u, v, w) \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with differential matrix

$$Df(x) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

Show that the 2×2 minor $L(Df) := \det \begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} = v_y w_z - v_z w_y$ is a null-Lagrangian.

Solution 1. It is sufficiently simple to compute the Euler-Lagrange equations for the expression $L(Df) = v_y w_z - v_z w_y$. We obtain that

$$\begin{aligned}
&-\nabla \cdot D_{P^1} L(Df) + D_{z^1} L(Df) = 0 \\
&-\nabla \cdot D_{P^2} L(Df) + D_{z^2} L(Df) = -\nabla \cdot (0, w_z, -w_y) + 0 = -w_{zy} + w_{yz} = 0 \\
&-\nabla \cdot D_{P^3} L(Df) + D_{z^3} L(Df) = -\nabla \cdot (0, -v_z, v_y) + 0 = v_{zy} - v_{yz} = 0
\end{aligned}$$

There is no dependence of z (the variable in whose place you put the function f) in $L(Df)$, so the derivatives $D_{z^i} L(Df)$ vanish above. The expression $D_{P^i} L(Df)$ denotes a gradient of the function

$$L(Df) = L \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

with respect to the variables on the row i. Since the Euler-Lagrange equations are always satisfied, L is a null Lagrangian.

Exercise 2. [Evans, Problem 8.7.7] Prove that $L(P) := trace(P^2) - trace(P)^2$ is a null Lagrangian. Here the trace of an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n$ is defined $trace(A) = \sum_{j=1}^n a_{jj}$.

Solution 2. Let us first expand the formula, denoting $P = (p_{ij})$:

$$\operatorname{tr}(P^2) - \operatorname{tr}(P)^2 = \sum_{i,j=1}^n p_{ij} p_{ji} - \left(\sum_{i=1}^n p_{ii} \right)^2 = \sum_{i,j=1}^n p_{ij} p_{ji} - p_{ii} p_{jj}.$$

The expression $p_{ij} p_{ji} - p_{ii} p_{jj}$ is a 2×2 subdeterminant of the matrix $\begin{pmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{pmatrix}$ obtained from the matrix P by removing all rows and columns except i and j. It happens that each of these subdeterminants is a null Lagrangian, much to the same reason as why the expression of Exercise 1 was one (in fact, subdeterminants are always null Lagrangians). To prove this, we compute the Euler-Lagrange equations for $L_{ij} = p_{ij}p_{ji} - p_{ii}p_{jj}$ as

$$-\nabla \cdot D_{P^k} L_{ij}(Df) + D_{z^k} L_{ij}(Df) = 0, \qquad \text{when } k \neq i, j$$
$$-\nabla \cdot D_{P^i} L_{ij}(Df) + D_{z^i} L_{ij}(Df) = -f^j_{z^j z^i} + f^j_{z^i z^j} = 0$$
$$-\nabla \cdot D_{P^j} L_{ij}(Df) + D_{z^j} L_{ij}(Df) = f^i_{z^j z^i} - f^i_{z^i z^j} = 0.$$

Exercise 3. [Evans, Problem 8.7.4] Assume $\eta : \mathbb{R}^n \to \mathbb{R}$ is C^1.

(i) Show that $L(P, z, x) := \eta(z) \det P$ is a null Lagrangian; here $P \in \mathbb{M}^{n\times n}$, $z \in \mathbb{R}^n$.

(ii) Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, then

$$\int_\Omega \eta(f) \det(Df) dx$$

depends only on $f_{|\partial\Omega}$.

Solution 3. a) We compute again by Euler-Lagrange equations.

$$\begin{aligned} -\nabla_x \cdot D_{P^k} L(Df, f) + D_{z^k} L(Df, f) \\ &= -\nabla_x \cdot (\eta(f) D_{P^k} \det Df) + (D_{z^k}\eta)(f) \det Df \\ &= -\eta(f)\nabla_x \cdot D_{P^k} \det Df - \nabla_x \eta(f) \cdot D_{P^k} \det Df + (D_{z^k}\eta)(f) \det Df \end{aligned}$$

The first term is just $\eta(f)$ times the Euler-Lagrange equation for the Jacobian $\det Df$. The Jacobian is known to be a null Lagrangian, so we do not repeat the proof here. One may compute

$$\nabla_x \eta(f) = \left(\sum_{j=1}^n \eta_{z^j}(f) f^j_i \right)_{i=1}^n$$

We use the cofactor expansion for the determinant with row k:

$$\det Df = \sum_{i=1}^n (-1)^{i+k} f^k_i M_{ki},$$

where M_{ki} denotes the determinant of the matrix we get by removing row k and column i from Df. Thus

$$D_{P^k} \det Df = \left((-1)^{i+k} M_{ki}\right)_{i=1}^n .$$

This finally gives

$$-\nabla_x \eta(f) \cdot D_{P^k} \det Df = -\sum_{i=1}^n \sum_{j=1}^n \eta_{z^j}(f) f^j_i (-1)^{i+k} M_{ki}$$

Note that the term in the above sum with $j = k$ is exactly $(D_{z^k}\eta)(f)\det Df$, which cancels out the similar term in the Euler-Lagrange equation. The rest is equal to

$$-\sum_{j\neq k}\eta_{z^j}(f)\sum_{i=1}^{n} f_i^j(-1)^{i+k}M_{ki}$$

We now expand each subdeterminant $(\operatorname{cof} Df)_{ki}$ with respect to the jth row, which gives

$$(\operatorname{cof} Df)_{ki} = \sum_{l\neq i}(-1)^{l+j}(-1)^{\chi(l>i)+\chi(j>k)} f_l^j M_{ki,jl},$$

where $M_{ki,jl}$ denotes the determinant of the matrix we get by removing rows k and j and columns i and l from Df. Here also $\chi(a > b)$ is equal to 1 if $a > b$ and 0 otherwise. The factor $(-1)^{\chi(l>i)+\chi(j>k)}$ comes from the fact that when we remove row k and column i, we have to swap all the $\pm$-signs that come after. Thus what remains of the Euler-Lagrange equation reads

$$-\sum_{j\neq k}\eta_{z^j}(f)\sum_{i=1}^{n}\sum_{l\neq i}(-1)^{i+k+l+j}(-1)^{\chi(l>i)+\chi(j>k)} f_l^j f_i^j M_{ki,jl}.$$

Obviously $M_{ki,jl} = M_{kl,ji}$. But this means that in the last two sums, the terms (i, l) and (l, i) cancel each other out because of the factor $(-1)^{\chi(l>i)}$. This shows that the whole expression is zero, and hence that L is a null Lagrangian.

b) Follows from the alternate characterization of null Lagrangians, and the fact that the above computation may be generalized to $f \in C^1$ in the weak sense.

Exercise 4. [Evans, Problem 8.7.5] If $f : \mathbb{R}^n \to \mathbb{R}^n$ is as in Problem 3, fix $x_0 \notin f(\partial\Omega)$. If r is so small that $B(x_0, r) \cap f(\partial\Omega) = \emptyset$, choose a C^1-map η so that $\int_{\mathbb{R}^n}\eta(z)dz = 1$ and $\eta(x) = 0$ when $|x - x_0| \geq r$.

Define

$$deg(f, x_0) = \int_\Omega \eta(f)\det(Df)dx,$$

the *degree* of f relative to x_0. Prove that the degree is an integer.

Solution 4. Solution will be added a bit later.

Exercise 5. In geometric function theory one studies the *distortion* of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Writing $f = (u, v)$ and assuming that the Jacobian $\det(Df(x)) > 0$ is positive almost everywhere, the distortion is defined by

$$K(f) := \frac{|\partial_x u|^2 + |\partial_y u|^2 + |\partial_x v|^2 + |\partial_y v|^2}{\det(Df)}$$

Show that the functional $L(Df) := K(f)$ is polyconvex; do this by first showing that $F(x,y) = x^2/y$ is convex on $(0,\infty)\times(0,\infty)$.

[Hint: You need to show that $F(x,y) - F(a,b) \geq 2ab^{-1}(x-a) - ab^{-2}(y-b)$]

Note. In higher dimensions the distortion of a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$K(f) := \frac{\left[\sum_{j,k=1}^{n} |\partial_{x_j} f^k|^2\right]^{n/2}}{\det(Df)}$$

so that $K(tf) = K(f)$ for all $t \in \mathbb{R}$. Also in higher dimensions the distortion is polyconvex, but the algebra to prove this is a little more difficult.

Solution 5. Let us first show that $F(x,y) = x^2/y$ is convex as a function of two real variables. Let $0 < t < 1$. We want to prove that

$$F(tx + (1-t)a, ty + (1-t)b) \leq tF(x,y) + (1-t)F(a,b)$$

This reduces to

$$\begin{aligned}
\Leftrightarrow \quad & \frac{t^2x^2 + 2t(1-t)ax + (1-t)^2a^2}{ty + (1-t)b} \ \leq \ \frac{tx^2}{y} + \frac{(1-t)a^2}{b} \\
\Leftrightarrow \quad & t^2x^2yb + 2t(1-t)axyb + (1-t)^2a^2yb \ \leq \ (ty + (1-t)b)(tx^2b + (1-t)a^2y) \\
\Leftrightarrow \quad & 2t(1-t)axyb \ \leq \ t(1-t)(x^2b^2 + a^2y^2) \\
\Leftrightarrow \quad & 0 \ \leq \ t(1-t)(xb - ay)^2.
\end{aligned}$$

Thus our expression is convex. Now let us consider the distortion as a function

$$K(P,r) = \frac{p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2}{r} = \frac{|P|^2}{r}.$$

Here $|P| = (p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2)^{1/2}$, and we remark that $|P|^2$ is a convex function of the matrix P because the function $f(x) = x^2$ is convex as well. Then if $0 < t < 1$,

$$\begin{aligned}
K(tP_1 + (1-t)P_2, tr_1 + (1-t)r_2) &= \frac{|tP_1 + (1-t)P_2|^2}{tr_1 + (1-t)r_2} \\
&\leq \frac{t|P_1|^2 + (1-t)|P_2|^2}{tr_1 + (1-t)r_2} \\
&\leq \frac{t|P_1|^2}{r_1} + \frac{(1-t)|P_2|^2}{r_2} \\
&= tK(P_1,r_1) + (1-t)K(P_2,r_2).
\end{aligned}$$

This proves the polyconvexity.