SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 1

Exercise 1. Suppose $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary. Assume also that $A(x) = (a_{i,j}(x))$ is symmetric and uniformly elliptic, so that $\lambda|\xi|^2 \le \xi \cdot A(x)\xi \le \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$.

We know from the lectures that the variational integral

$$I(u) = \int_\Omega Du(x) \cdot A(x)Du(x) + f(x)u(x)\, dx$$

has a minimizer in the set $\mathcal{A}(g) := \{v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega)\}$. Prove that $I(\frac{1}{2}(u+v)) < \frac{1}{2}I(u) + \frac{1}{2}I(v)$ for $u, v \in W^{1,2}(\Omega)$ unless $u = v$ almost everywhere, and use this to show that the minimiser is unique.

Solution 1. We first prove the pointwise inequality

$$\left\langle \frac{1}{2}(Du + Dv), A(x)\frac{1}{2}(Du + Dv) \right\rangle \le \frac{1}{2}\langle Du, A(x)Du\rangle + \frac{1}{2}\langle Dv, A(x)Dv\rangle.$$

After some simplification, this reads

$$0 \le \frac{1}{4}\langle Du - Dv, A(x)(Du - Dv)\rangle,$$

which is true thanks to the ellipticity condition for A. Equality holds if $Du(x) = Dv(x)$. This also implies that

$$I\left(\frac{1}{2}(u+v)\right) \le \frac{1}{2}I(u) + \frac{1}{2}I(v),$$

since the term "$+f(x)u(x)$" is linear. Equality holds iff $Du = Dv$ almost everywhere, which implies $u = v$ almost everywhere if u and v have the same boundary values, say $u, v \in \mathcal{A}(g)$. This implies the uniqueness of minimizers to $I(u)$, since if $u, v \in \mathcal{A}(g)$ are distinct minimizers, then the function $\frac{1}{2}(u+v) \in \mathcal{A}(g)$ would have even lower energy than u and v.

Exercise 2. [Evans, Problem 8.7.8] Explain why the methods studied in the lectures, i.e. Evans Chapter 8.2, will *not* work for the integral representing the area of the graph of a function,

$$I(w) = \int_\Omega \left(1 + |Dw|^2\right)^{1/2} dx,$$

over $\mathcal{A}(g) = \{w \in W^{1,2}(\Omega) : w - g \in W_0^{1,2}(\Omega)\}$ for any $1 \le q < \infty$.

Solution 2. The issue is with the coercivity condition in 8.2.1. Evans requires that

$$L(p, z, x) \geq \alpha |p|^q - \beta$$

for some $q > 1$. This is not the case when $L(p, z, x) = (1 + |p|^2)^{1/2}$. One might also ask whether it would be enough to assume the coercivity condition for $q = 1$. If one follows Evans' proof, this is not enough. The reason is that the coercivity condition is used to establish the weak convergence of a minimizing sequence in the space $\mathcal{W}^{1,q}$. As we have seen in Exercise 5 of set 6, a bounded sequence in L^1 need not have a subsequence converging weakly. Thus in Evans' proof it is essential that the coercivity condition holds for $q > 1$.

Exercise 3. Given $g \in W^{1,2}(\Omega)$ show that the Dirichlet problem

$$\begin{cases} -\Delta u + u^3 = 0, \\ u - g \in W_0^{1,2}(\Omega) \end{cases}$$

has at least one weak solution $u \in W^{1,2}(\Omega)$, if $\Omega \subset \mathbb{R}^4$ is bounded with C^1-boundary.

[Hint: Express u as a solution to the Euler-Lagrange equation of a suitable variational integral.]

Solution 3. We guess the variational integral that the equation comes from first. The equation is Laplace's equation with an added term, so it is not hard to guess that we should choose

$$L(Du, u, x) = \frac{1}{2}|Du|^2 + \frac{1}{4}u^4.$$

Indeed, the Euler-Lagrange equation for this functional is $-\Delta u + u^3 = 0$. Thus to prove that the Euler-Lagrange equation has at least one weak solution, it is enough to prove that

$$I(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 + \frac{1}{4}u^4 \right) dx$$

has a minimizer in the class $\mathcal{A}(g) = g + \mathcal{W}_0^{1,2}(\Omega)$. First, though, we should check whether it makes sense to integrate u^4, as we only assumed $u \in \mathcal{W}^{1,2}$. This is okay because of the Sobolev embedding $\mathcal{W}^{1,2}(\Omega) \subset L^4(\Omega)$, as we are in the dimension $n = 4$.

We use Theorem 2 from section 8.2 in Evans to find a minimizer. We have to check coercivity and convexity of $L(p, z, x) = \frac{1}{2}|p|^2 + \frac{1}{4}z^4$. The convexity in the variable p is clear, since the expression $|p|^2$ is the same as for the Dirichlet energy. The coercivity is also easy, we find that

$$\frac{1}{2}|p|^2 + \frac{1}{4}z^4 \geq \frac{1}{2}|p|^2.$$

These combined with Evans' theorem prove our claim.

Exercise 4. If $a, b \in \mathbb{R}$ and $0 < t < 1$, define $w : \mathbb{R} \to \mathbb{R}$ by

$$w(s) = \begin{cases} as, & \text{if} \quad 0 \le s < t, \\ bs + t(a-b), & \text{if} \quad t \le s \le 1, \\ w(s-n) + nw(1), & \text{if} \quad n < s \le n+1, \quad n \in \mathbb{N}. \end{cases}$$

Given $0 \neq x_0 \in \mathbb{R}^n$ let then $u_k(x) = w(kx \cdot x_0)/k$.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, show that the sequence $u_k(x) := w(kx) \in W^{1,q}(\Omega)$, for every $1 \le q \le \infty$ and $k \in \mathbb{N}$. Furthermore, show that for $1 < q < \infty$ the sequence $\{u_k\}_{k \in \mathbb{N}}$ converges weakly in $W^{1,q}(\Omega)$, and *determine* its weak limit $u \in W^{1,q}(\Omega)$.

[Hint: Draw the graph of $w(s)$ and recall Problem 2 in Exercises 7]

Solution 4. Firstly, the fact that $u_k(x) \in \mathcal{W}^{1,q}(\Omega)$ for every $q \in [1, \infty]$ follows immediately from the fact that u_k is locally affine. This implies that both u_k and its derivatives are locally bounded (the derivatives are even piecewise constant), which gives the required Sobolev-regularity.

We would next like to find the weak limit of the sequence u_k in $\mathcal{W}^{1,q}$ for $q \in (1, \infty)$. The proof will be essentially the same as in Exercise 2 of set 7, so we will omit some details. The essential fact to know is that it is enough to prove the weak convergence of u_k in $L^q(\Omega)$, as well as the weak convergence of the derivatives in $L^q(\Omega)$. Let us show that these weak convergences hold first.

Let $\tilde{w}(s) = (ta + (1-t)b)s$, so that $\tilde{w}$ is a linear function. Define $u(x) = \tilde{w}(x \cdot x_0)$. Then u is an affine function, with derivative equal to $ta + (1-t)b$ in the direction x_0 and zero in the orthogonal directions. We claim that u will be the weak limit of the u_k.

The proof that the derivatives (any directional derivatives) of u_k converge weakly to u is the same as in Exercise 2 of set 7. For example, in the x_0 direction the derivatives oscillate rapidly between the values a and b, and thus eventually converge weakly to the weighted average $ta + (1-t)b$ when we test them with smooth functions. The proof of the fact that $u_k \to u$ in $L^q(\Omega)$ is similar too: We test with smooth functions first and see that the oscillation of u_k eventually averages out to an affine function u.

Let us now try to conclude why $u_k \to u$ weakly in $\mathcal{W}^{1,q}(\Omega)$. We use the classification of the dual of $\mathcal{W}^{1,q}$ for $1 < q < \infty$, mentioned shortly in the end of Section 5 of Evans' book in the case $q = 2$. Any functional $v \in \mathcal{W}^{1,q}(\Omega)^*$ may be written in the form

$$\langle v, g \rangle = \int_\Omega f_0(x) g(x) dx + \sum_{j=1}^n \int_\Omega f_j(x) g_{x_j}(x) dx,$$

where $g \in L^q(\Omega)$ and $f_0, \ldots, f_n \in L^{q^*}(\Omega)$. The proof of this fact follows the same lines as in Evans' book and as discussed in the lectures. First we embed $\mathcal{W}^{1,q}(\Omega)$ into $L^q(\Omega) \times L^q(\Omega)^n$, then we extend any linear functional on $\mathcal{W}^{1,q}(\Omega)^*$ into the dual of $L^q(\Omega)\times L^q(\Omega)^n$ by Hahn-Banach. The dual of $L^q(\Omega) \times L^q(\Omega)^n$ is $L^{q^*}(\Omega) \times L^{q^*}(\Omega)^n$, which gives the formula above. Thus $u_k \to u$ weakly in $\mathcal{W}^{1,q}(\Omega)$.

Exercise 5. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is *not* convex, so that there are $z_0, y_0 \in \mathbb{R}^n$ and $0 < t < 1$ so that $F(tz_0 + (1-t)y_0) > tF(z_0) + (1-t)F(y_0)$ and assume that F is bounded. Let $\Omega = B(0,1)$ be the unit ball of $\mathbb{R}^n$.

Show that the variational integral

$$I(u) = \int_\Omega F(Du)\,dx$$

is *not* weakly lower semicontinuous in any $W^{1,q}(\Omega)$, $1 < q < \infty$.

[Hint: Consider first the case $tz_0 + (1-t)y_0 = 0$; use here Problem 4]

Solution 5. Suppose first that $tz_0 + (1-t)y_0 = 0$. Thus $z_0 = ax_0$ and $y_0 = bx_0$ for some $a, b \in \mathbb{R}$. We now consider the sequence u_k as defined in Exercise 4. If one computes the gradient of u_k, we find that

$$Du_k(x) = ax_0 \quad \text{or} \quad Du_k(x) = bx_0 \quad \text{almost everywhere.}$$

For the (weak) limit function u we have $Du(x) = (ta + (1-t)b)x_0$ everywhere. Thus

$$\int_\Omega F(Du(x))dx = |\Omega|\, F((ta + (1-t)b)x_0)$$

and

$$\int_\Omega F(Du_k(x))dx = t|\Omega|\, F(ax_0) + (1-t)|\Omega|\, F(bx_0),$$

since the sets where $Du_k(x)$ takes the values ax_0 and bx_0 are of size $t|\Omega|$ and $(1-t)|\Omega|$ respectively. This proves the claim, since now we have that

$$\int_\Omega F(Du(x))dx > \lim_{k\to\infty} \int_\Omega F(Du_k(x))dx.$$

Consider now the case $tz_0 + (1-t)y_0 = v_0 \neq 0$. Thus $z_0 - v_0 = ax_0$ and $y_0 - v_0 = bx_0$. Define u_k as before, but consider instead the sequence $v_0 \cdot x + u_k(x)$, which converges weakly to $v_0 \cdot x + u(x)$. Plugging these instead in the variational integral gives the desired result.