SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 7

Exercise 1. a) Derive the Euler-Lagrange equations for the variational integral

$$I(u) = \int_\Omega F\big(Du(x)\big)dx,$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

b) Consider the variational integral $I(u) = \int_{-1}^{1}[x\,u'(x)]^2 dx$ from the counterexample of Weierstrass [c.f. the notes/Section 12, on homepage].

Find the solutions to the Euler-Lagrange equation of this variational integral.
[Hint: Non-constant solutions will have a singularity at $x = 0$.]

Solution 1. a) We refer the reader to the derivation in Evans' book, pages 432-434. The end result (in our case) is the equation

$$\sum_{i,j=1}^{n} F_{p_i p_j}(Du)u_{x_i x_j} = 0.$$

b) In this case, the Euler-Lagrange equations read

$$(x^2 u'(x))' = 0.$$

Equivalently

$$x^2 u'(x) = C, \quad \Leftrightarrow \quad u(x) = A/x + B.$$

However, one should note that if $A \neq 0$ this function is not in $\mathcal{W}^{1,2}$.

Exercise 2. [Evans 8.6.1.b] Consider weakly converging sequences $(u_k)_{k=1}^{\infty} \subset L^p(0,1)$, where $1 < p < \infty$; see notes/Section 8, on homepage.

If $a, b \in \mathbb{R}$ and $0 < \lambda < 1$, let

$$u_k(x) \;=\; \begin{cases} a, & \text{if} \quad j/k \le x < \lambda(j+1)/k, \\ b, & \text{if} \quad \lambda(j+1)/k \le x < (j+1)/k. \end{cases} \qquad (j = 0, \dots, k-1)$$

[Draw a picture] Show that $(u_k)_{k=1}^{\infty}$ converges weakly to $u(x) \equiv \lambda a + (1-\lambda)b$ in $L^p(0,1)$.

Solution 2. We need to show that

$$\lim_{k\to\infty} \int_0^1 u_k(x)g(x)dx = \int_0^1 u(x)g(x)dx$$

for all $g \in L^q(0,1)$. It is enough to prove this for $g \in C_0^\infty(0,1)$, since they are dense in $L^q(0,1)$. The functions u_k alternate between the values a and b rapidly. The intervals on which u_k takes the value a are of length λ/k and on which it takes the value b have length $(1-\lambda)/k$. The smoothness of g will imply that this averages out to taking the integral of a weighted average $\lambda a+(1-\lambda)b$ of a and b multiplied by g. We give a short proof of this.

We may as well assume that $b=0$, otherwise replace u_k by u_k-b. Denote by $U_\lambda = U_\lambda(k)$ the union of the intervals on which $u_k = a$. We wish to show that

$$\lim_{k\to\infty}\int_{U_\lambda} g(x)dx = \lambda\int_0^1 g(x)dx.$$

To prove this, split the interval $[0,1]$ into subintervals $I_j = [j/k,(j+1)/k]$. On each such subinterval, make the change of variables

$$y = j/k + \lambda(x - j/k),$$

essentially squeezing each interval to an interval of length λ/k with the same left endpoint. In this way, we see that

$$\lambda\int_0^1 g(x)dx = \sum_{j=1}^{k}\int_{I_j} g(\lambda^{-1}(y-j/k)+j/k)dy.$$

Meanwhile,

$$\int_{U_\lambda} g(x)dx = \sum_{j=1}^{k}\int_{I_j} g(y)dy.$$

Now, the distance between the numbers $\lambda^{-1}(y-j/k)+j/k$ and y is at most $1/k$. Thus by uniform continuity of g we see that the above two expressions must converge to each other as $k\to\infty$. This finishes the proof.

Exercise 3. [Evans 8.6.2] Find $L = L(p,z,x)$ so that the PDE

$$-\Delta u + D\phi\cdot Du = f \qquad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to the functional $I(w) = \int_\Omega L\big(Dw,w,x\big)dx$. Here ϕ, f are given functions smooth in $\overline{\Omega}$.

Solution 3. Consider first L of the form

$$L(Du,u,x) = \psi(x)|Du|^2.$$

Then the Euler-Lagrange equations are of the form

$$-\sum_{j=1}^{n}(\psi(x)u_{x_i})_{x_i} = -\psi\Delta u - D\psi\cdot Du = 0.$$

In order for this to be equivalent to the original equation, we must have that

$$\psi_{x_i} = -\phi_{x_i}\psi$$

for every j. This is satisfied by choosing $\psi = e^{-\phi}$. Thus $L = e^{-\phi}|Du|^2$.

Exercise 4. [Evans 8.6.3] The *elliptic regularisation* of the heat equation is the PDE

$$(*) \qquad \partial_t u - \Delta u - \varepsilon\,\partial_t^2 u = 0 \qquad \text{in} \frac{1}{2}\,\Omega_T,$$

where $\varepsilon > 0$, $\Omega_T = \Omega \times (0, T]$ and $\Omega \subset \mathbb{R}^n$. Show that (*) is the Euler-Lagrange equation corresponding to an energy integral

$$I_\varepsilon(w) = \int_{\Omega_T} L_\varepsilon\big(Dw, \partial_t w, w, x, t\big)\,dxdt.$$

[Here $Du = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ is the space gradient of u]

Solution 4. The $\epsilon\partial_t^2 u$ term suggests that we should try

$$L = \psi(|Du|^2 + \epsilon|\partial_t u|^2).$$

This gives the equation

$$-\sum_{j=1}^{n} (\psi u_{x_i})_{x_i} - \epsilon(\psi\partial_t u)_t = 0.$$

Let us look for ψ that only depend on the variable t. In this case the equation above takes the form

$$-\psi(t)\Delta u - \epsilon\psi(t)\partial_t^2 u - \epsilon\psi'(t)\partial_t u = 0.$$

Thus we must choose ψ such that $\epsilon\psi'(t) = -\psi(t)$, giving $\psi(t) = e^{-t/\epsilon}$. Thus $L = e^{-t/\epsilon}(|Du|^2 + \epsilon|\partial_t u|^2)$.

Exercise 5. [Evans 6.6.2] A function $u \in W_0^{2,2}(\Omega) = H_0^2(\Omega)$ is a weak solution of the following boundary value problem for the *biharmonic equation*

$$(1) \qquad \begin{cases} \Delta^2 u \;=\; f, & \text{in } \Omega, \\ u \;=\; \frac{\partial u}{\partial \nu} \;=\; 0, & \text{on } \partial\Omega, \end{cases}$$

provided

$$\int_\Omega \Delta u\,\Delta v\,dx \;=\; \int_\Omega f\,v\,dx \qquad \forall v \in W_0^{2,2}(\Omega).$$

Given $f \in L^2(\Omega)$, prove that there always exists a weak solution to (1).

Solution 5. We apply the Lax-Milgram theorem. Choose $H = \mathcal{W}_0^{2,2}(\Omega)$ as the Hilbert space. In particular, the boundary condition $u = \frac{\partial u}{\partial \nu} = 0$ is satisfied for any $u \in H$. We only need to find $u \in H$ such that

$$B(u, v) = \int_\Omega f\,v\,dx, \quad \text{where} \quad B(u, v) = \int_\Omega \Delta u\,\Delta v\,dx.$$

By Cauchy-Schwarz, we have

$$|B(u,v)| \leq ||u||_H ||v||_H.$$

For the opposite direction, we may apply the Poincare inequality to get

$$||u||_{L^2(\Omega)} \leq C||D^2u||_{L^2(\Omega)} \quad \text{and} \quad ||Du||_{L^2(\Omega)} \leq C||D^2u||_{L^2(\Omega)}.$$

Thus to prove that

$$||u||_H^2 \leq c|B(u,u)| \text{ for some } c > 0,$$

we only need to establish an inequality of the form $||D^2u||_{L^2(\Omega)} \leq C||\Delta u||_{L^2(\Omega)}$. We claim, in fact, that

$$||D^2u||_{L^2(\Omega)} = ||\Delta u||_{L^2(\Omega)}, \quad u \in \mathcal{W}_0^{2,2}(\Omega).$$

By approximation, it is enough to verify this for smooth u. We compute that

$$\begin{aligned}
\int_\Omega |D^2u|^2 dx &= \sum_{i,j=1}^n \int_\Omega u_{x_i x_j} u_{x_i x_j} dx \\
&= \sum_{i,j=1}^n \int_\Omega u u_{x_i x_j x_i x_j} dx \\
&= \sum_{i,j=1}^n \int_\Omega u_{x_i x_i} u_{x_j x_j} dx \\
&= \int_\Omega \left(\sum_{j=1}^n u_{x_j x_j} \right)^2 dx \\
&= \int_\Omega |\Delta u|^2 dx.
\end{aligned}$$

Applying Lax-Milgram with the functional $\langle f, v \rangle = \int_\Omega fv dx$ finishes the proof.