SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 6

Exercise 1. Recall the continuous linear operators $T : X \to Y$ between Banach spaces X and Y; and that these have the norm $\|T\| := \sup\{\|Tx\| \,:\, \|x\| \leq 1\}$.

If $T_k : X \to Y$ are compact linear operators and $\|T - T_k\| \to 0$, show that $T : X \to Y$ is compact.

[Hint: Recall the different characterisations of compactness in Banach spaces]

Solution 1. Recall the following notion of precompactness: A set is precompact iff for every $\epsilon > 0$ it admits a finite cover of balls with radius ϵ.

We prove now that $T(B)$ is precompact, where B is the unit ball in X. Let $\epsilon > 0$. Take m so large that $||T - T_m|| < \epsilon/2$, and choose by compactness a finite cover of $T_m(B)$ with balls of radius $\epsilon/2$. Let the centers of these balls be $y_1, \ldots, y_M$. It is enough to prove that the balls $B(y_1, \epsilon), \ldots, B(y_M, \epsilon)$ cover $T(B)$. Let $x \in B$. Then $T_m(x) \in T_m(y_j, \epsilon/2)$ for some j. Thus

$$||T(x) - y_j||_Y \leq ||T(x) - T_m(x)||_Y + ||T_m(x) - y_j||_Y < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves the claim.

Exercise 2. Let $B = B(0, 1) \subset \mathbb{R}^2$. Then, as will be discussed later,

$$u(x) := (Tf)(x) = \int_B \log |x - y| f(y) dy$$

is a solution to the Poisson equation $\Delta u = f$. Show that for $2 < p < \infty$, $T : L^p(B) \to W^{1,p}(B)$ is a continuous linear operator. Deduce that Tï¿½is compact as an operator $T : L^p(B) \to L^p(B)$.

Solution 2. Will be added later.

Exercise 3. Suppose $u \in W^{1,p}(\Omega)$, for some $1 < p < \infty$. If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous with $f(0) = 0$, use difference quotients to show that $f \circ u \in W^{1,p}(\Omega)$.

This is a (strong !) generalisation of Problem 4/Exercises 2. As an application, show that the positive part $u^+ \in W^{1,p}(\Omega)$; here $u^+(x) = u(x)$ if $u(x) \geq 0$ and $u^+(x) = 0$ otherwise.

Solution 3. We use Theorem 3 from Evans' book, section 5.8. Suppose that $u \in \mathcal{W}^{1,p}(\Omega)$. Then by Evans' theorem, we have the following bound for the difference quotients

$$||D_h u||_{L^p(V)} \leq C||Du||_{L^p(\Omega)},$$

where one can check that the constant C does not depend on the compact subset $V \subset\subset \Omega$. If L is the Lipschitz constant of f, then we may now estimate that

$$\left| \frac{f(u(x+he_j)) - f(u(x))}{h} \right| \leq L \left| \frac{u(x+he_j) - u(x)}{h} \right|,$$

and thus

$$||D_h(f \circ u)||_{L^p(V)} \leq L||D_h u||_{L^p(V)} \leq C_1||Du||_{L^p(\Omega)}.$$

Letting $V \to \Omega$ gives that $D(f \circ u) \in L^p(\Omega)$. The estimate

$$|f(u(x))| = |f(u(x)) - f(0)| \leq L|u(x)|$$

also gives that $f \circ u \in L^p(\Omega)$. Thus $f \circ u \in \mathcal{W}^{1,p}(\Omega)$. Applying this result to the Lipschitz function $f(x) = \max(x, 0)$ proves the second part of the exercise too.

Exercise 4. Suppose $1 < s \leq p < \infty$ and $|\Omega| < \infty$, so that $L^p(\Omega) \subset L^s(\Omega)$. If $\|f_k\|_{L^p(\Omega)} \leq 1$, $k = 1, 2, \ldots$ and if $f_k \to f$ weakly in $L^s(\Omega)$, show that

$$f \in L^p(\Omega) \qquad \text{and} \qquad \|f\|_{L^p(\Omega)} \leq 1.$$

[Hint: Recall the $L^p - L^q$ duality; c.f. proof of "Lemma on weak limits in $L^p(\Omega)$" in notes on course web-page]

Solution 4. We denote the Hölder-conjugates of p and s by p' and s' respectively. Recall from the duality of L^p spaces that

$$||f||_{L^p} = \sup \left\{ \left| \int_\Omega fg\, dx \right| : ||g||_{L^{p'}} \leq 1 \right\}.$$

Let $g \in L^{s'}$ be such that $||g||_{L^{p'}} \leq 1$. Then by weak convergence,

$$\left| \int_\Omega fg\, dx \right| = \left| \lim_{k \to \infty} \int_\Omega f_k g\, dx \right| \leq ||f_k||_p \leq 1.$$

Now since $s \leq p$, we have $s' \geq p'$ and thus $L^{s'} \subset L^{p'}$. The above inequality proves that

$$\sup \left\{ \left| \int_\Omega fg\, dx \right| : g \in L^{s'}, ||g||_{L^{p'}} \leq 1 \right\} \leq 1.$$

However, $L^{s'}$ is dense in $L^{p'}$, which shows that we must also have

$$\sup \left\{ \left| \int_\Omega fg\, dx \right| : ||g||_{L^{p'}} \leq 1 \right\} \leq 1.$$

This concludes the proof that $||f||_{L^p} \leq 1$.

Exercise 5. (Evans, problem 5.10.11) Recall the difference quotients $D_j^h u(x)$ and the difference gradient $D^h u(x) = (D_1^h u(x), D_2^h u(x), \dots, D_n^h u(x))$.

Prove that Theorem 3 in Evans/Section 5.8 does not hold at $p = 1$: That is, show by an example that if we have $\|D^h u\|_{L^1(\Omega')} \leq C$ for all $\Omega' \subset\subset \Omega$ and for all $|h| \leq$dist$(\Omega', \partial\Omega)$), it does not necessarily hold that $u \in W^{1,1}(\Omega)$.

Solution 5. Note that the statement of this Exercise differs quite a bit from the actual Problem 11 in Evans, as Evans doesn't require the "for all $\Omega' \subset\subset \Omega$". Nevertheless, our counterexample will be local so it solves both questions.

For the counterexample, choose $u(x)$ as the characteristic function of some set $V \subset\subset \Omega$. It's enough if the set V has C^1-boundary, so a ball for example. Then $D_h u$ will be bounded in the L^1-norm uniformly in h. This is because the difference quotient

$$\frac{u(x + he_j) - u(x)}{h}$$

may only attain the values $\pm 1/h$ and 0. The set in which it attains the values $\pm 1/h$ is contained in the set

$$\{x \in \Omega : \text{dist}(x, \partial V) \leq h\}.$$

The above set has measure at most Ch for some constant C. Thus

$$||D_h u||_{L^1(\Omega)} \leq \int_{\text{dist}(x,\partial V) \leq h} \frac{1}{h} dx \leq C.$$

However, u is not in $\mathcal{W}^{1,1}(\Omega)$ even locally since it doesn't have proper weak derivatives.