SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 5

Exercise 1. a) Show that if $S : X \to Y$ and $T : X \to Y$ are compact operators, then $S + T : X \to Y$ is a compact operator.

b) If $T : X \to Y$ is a compact operator, and $S : Z \to X$, $R : Y \to W$ are continuous operators, for some Banach spaces Z and W, show that
$TS : Z \to Y$ and $RT : X \to W$ are compact operators.

Solution 1. From the definition one can see that an operator is compact if and only if it maps every bounded sequence to a sequence with a convergent subsequence.

a) If S and T are compact and $(x_n) \subset X$ is a bounded sequence, then (Sx_n) and (Tx_n) have convergent subsequences. Passing first to a subsequence in S and then to a subsequence in T we can choose a subsequence (x_{n_k}) so that both Sx_{n_k} and Tx_{n_k} converge. Thus $(S + T)x_{n_k}$ converges as well. It follows that $S + T$ is compact.

b) Let $(z_n) \subset Z$ be a bounded sequence. Then $(x_n) = (Sz_n)$ is also a bounded sequence since S is bounded. Thus (TSz_n) has a convergent subsequence, hence TS is compact. Suppose now that $(x_n) \subset X$ is a bounded sequence. Then (Tx_n) has a convergent subsequence (Tx_{n_k}). Thus RTx_{n_k} also converges due to continuity of R. Hence RT is compact.

Exercise 2. Suppose $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded subdomain.

a) Show that we have the compact embedding

$$W_0^{1,p}(\Omega) \subset\subset L^p(\Omega), \qquad 1 \le p < \infty.$$

b) If $\phi \in C_c^\infty(\Omega)$ is given, show that

$$T : W^{1,p}(\Omega) \to L^p(\Omega), \qquad (Tu)(x) = \phi(x)u(x),$$

is a compact operator.

Solution 2. a) Assume first that we have the compact embedding

$$\mathcal{W}^{1,p}(\Omega) \subset\subset L^p(\Omega) \tag{1}$$

for all $p \in [1, \infty)$ whenever Ω is a C^1-domain.

Suppose now that Ω is not necessarily C^1. Exercise 1 of set 4 shows that we have the continuous embedding

$$\mathcal{W}_0^{1,p}(\Omega) \subset \mathcal{W}_0^{1,p}(\mathbb{R}^n)$$

via the "extension as zero". Let $\Omega \subset \Omega'$, where Ω' is a larger domain that is also C^1 (like a ball). Then by restricting ourselves from $\mathbb{R}^n$ to Ω' we clearly also have that

$$\mathcal{W}_0^{1,p}(\Omega) \subset \mathcal{W}^{1,p}(\Omega').$$

By our assumption, we have that $\mathcal{W}_0^{1,p}(\Omega) \subset\subset L^p(\Omega')$. Let now (f_n) be a bounded sequence in $\mathcal{W}_0^{1,p}(\Omega)$. Then their extensions are convergent in $L^p(\Omega')$, which implies that they are also convergent in $L^p(\Omega)$ since we have chosen the extensions so that $f_n(x) = 0$ for $x \notin \Omega$. This gives the desired embedding.

To prove (1), we only need to consider the cases $p > n$ and $p = n$, since $p < n$ comes from Rellich-Kondrachov. We sketch the proof. For $p > n$ we have by Morrey's inequality the continuous embedding

$$\mathcal{W}^{1,p}(\Omega) \subset C^\gamma(\Omega).$$

By Arzela-Ascoli we have the compact embedding

$$C^\gamma(\Omega) \subset\subset C(\Omega).$$

We also have the continuous embedding

$$C(\Omega) \subset L^\infty(\Omega) \subset L^p(\Omega).$$

Combining all of these gives that $\mathcal{W}^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

For $p = n$ we make the following arguments. Let $q \geq 1$ be close to n. By Rellich-Kondrachov,

$$\mathcal{W}^{1,q}(\Omega) \subset\subset L^{q'}(\Omega), \quad q' < q* = \frac{nq}{n-q}.$$

Note that $q* \to \infty$ as $q \to n$. Choose thus q so that $q* > n$ and choose $q' = n$. Since Ω is bounded, we can compare L^n and L^q norms to get the continuous embedding

$$\mathcal{W}^{1,n}(\Omega) \subset \mathcal{W}^{1,q}(\Omega).$$

Combining this with the above gives $\mathcal{W}^{1,n}(\Omega) \subset\subset L^n(\Omega)$.

b) If $\phi \in C_0^\infty(\Omega)$, then $\phi u \in \mathcal{W}_0^{1,p}(\Omega)$ for every $u \in \mathcal{W}^{1,p}(\Omega)$. We also have that $||\phi u||_{\mathcal{W}^{1,p}} \leq C||u||_{\mathcal{W}^{1,p}}$. Thus we have that

$$T : \mathcal{W}^{1,p}(\Omega) \to \mathcal{W}_0^{1,p}(\Omega) \subset\subset L^p(\Omega).$$

This clearly shows that $T : \mathcal{W}^{1,p}(\Omega) \to L^p(\Omega)$ must be compact.

Exercise 3. Suppose $f : [0,1] \to \mathbb{R}$ is continuous, and let $f_s(x) = f(sx)$ for $s, x \in [0,1]$. Determine whether the set $H = \{f_s : 0 \leq s \leq 1\}$ is relatively compact in the space $C[0,1] = \{g : [0,1] \to \mathbb{R} \text{ continuous}\}$.

Solution 3. The set is relatively compact for every f. Suppose we have a sequence $(f_{s_n}) \subset H$. Choose a subsequence of the real numbers $s_n \in [0,1]$ that converges, say to some $s \in [0,1]$. Then since f is uniformly continuous, for each $\epsilon > 0$ and $n > n_\epsilon$ we can make the following estimates:

$$\begin{aligned} |s - s_n| < \delta &\Leftrightarrow |sx - s_n x| < \delta, \quad \forall x \in [0,1] \\ &\Leftrightarrow |f(sx) - f(s_n x)| < \epsilon, \quad \forall x \in [0,1] \\ &= ||f_s - f_{s_n}||_\infty < \epsilon. \end{aligned}$$

Thus $f_{s_n} \to f_s$ in $C(0,1)$.

Exercise 4. If $K : [0,1] \times [0,1] \to \mathbb{C}$ is continuous, and $T : C[0,1] \to C[0,1]$ is given by

$$(Tf)(x) = \int_0^1 K(x,t) f(t) dt,$$

show that T is a compact operator.

Solution 4. Let B denote the unit ball in $C(0,1)$. We want to show that $T(B)$ is relatively compact. By Arzela-Ascoli it is enough to show that $T(B)$ is equicontinuous. We may estimate that

$$|(Tf)(x) - (Tf)(y)| = \left| \int_0^1 (K(x,t) - K(y,t)) f(t) dt \right| \leq ||f||_\infty \int_0^1 |K(x,t) - K(y,t)| \, dt.$$

This shows the equicontinuity of $T(B)$, since $||f||_\infty < 1$ and K is uniformly continuous on $[0,1]^2$.

Exercise 5. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary $\partial\Omega$. If

$$k < \frac{n}{p} \quad \text{and} \quad \frac{1}{q_0} = \frac{1}{p} - \frac{k}{n},$$

show that we have the compact embedding $W^{k,n}(\Omega) \subset\subset L^q(\Omega)$ for every $1 \leq q < q_0$.

Solution 5. There is a typo in the statement of the exercise. The embedding should read $\mathcal{W}^{k,p}(\Omega) \subset\subset L^q(\Omega)$.

We proceed by induction on k. Case $k = 1$ is the Rellich-Kondrachov compactness theorem. Assume the embedding holds for some k and all p, where $p < n/k$. Let now $p < n/(k+1)$. We would first like to make the compact embedding

$$\mathcal{W}^{k+1,p}(\Omega) \subset\subset \mathcal{W}^{k,q}(\Omega) \tag{2}$$

for some exponent q. We will use the Rellich-Kondrachov theorem to show that this embedding is valid for all $q < q*$, where

$$q* = \frac{np}{n-p}$$

We will first check how this implies that the next induction step is valid. Note that

$$q* < \frac{n \cdot \frac{n}{k+1}}{n - \frac{n}{k+1}} = \frac{n}{k}.$$

Thus we have by the induction assumption that

$$\mathcal{W}^{k,q*}(\Omega) \subset\subset L^{q'}(\Omega),$$

where $q' < q_0'$ with q_0' defined by

$$\frac{1}{q_0'} = \frac{1}{q*} - \frac{k}{n} = \frac{n-p}{np} - \frac{k}{n} = \frac{1}{p} - \frac{k+1}{n}.$$

Combining this with the embedding $\mathcal{W}^{k+1,p}(\Omega) \subset\subset \mathcal{W}^{k,q}(\Omega)$, $q < q*$ gives that

$$\mathcal{W}^{k+1,p}(\Omega) \subset\subset L^{q_0'}(\Omega),$$

which is exactly what we wanted since $1/q_0' = 1/p - (k+1)/n$.

To see why the embedding (2) holds, simply take a bounded sequence (f_n) in $\mathcal{W}^{k+1,p}(\Omega)$. Then the kth order partial derivatives of f_n are in $\mathcal{W}^{1,p}(\Omega)$, which embeds into $L^q(\Omega)$ compactly by Rellich-Kondrachov. By choosing enough subsequences we may assume that all of these kth order partial derivatives converge in $L^q(\Omega)$. To see that f_n converges in $\mathcal{W}^{k,q}(\Omega)$, we now only need to show that the derivatives of order up to $k-1$ also converge in $L^q(\Omega)$. But this is immediate from the continuous embedding $\mathcal{W}^{1,q}(\Omega) \subset L^q(\Omega)$ (valid since $q < q* < n/k \leq n$).