SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 3

Exercise 1. Suppose $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded subdomain. If $u \in W_0^{1,p}(\Omega)$, set

$$\overline{u}(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$$

Show that $\overline{u} \in W^{1,p}(\mathbb{R}^n)$.

Solution 1. The only issue here is to check whether the weak partial derivatives of $\overline{u}$ are well-defined in the whole space and if they can be calculated from the formula

$$\partial_{x_j}\overline{u}(x) = \begin{cases} \partial_{x_j}u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}.$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a test function. We have to verify that

$$\int_{\mathbb{R}^n} \overline{u}(x)\partial_{x_j}\phi(x)dx = -\int_\Omega \partial_{x_j}u(x)\phi(x)dx.$$

Let u_m be a sequence in $C_0^\infty(\Omega)$ tending to u in the Sobolev sense. Then by smoothness the above formula holds for each u_m. Going to the limit also gives the formula for u, especially since by definition

$$\int_{\mathbb{R}^n} \overline{u}(x)\partial_{x_j}\phi(x)dx = \int_\Omega u(x)\partial_{x_j}\phi(x)dx.$$

Exercise 2. Let $0 \le \eta \in C_c^\infty(\mathbb{R}^n)$, with $\text{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta(x)dx = 1$, be a standard mollifier and set $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$.

If $f \in L^1_{loc}(\mathbb{R}^n)$, show that we have

$$(\eta_\varepsilon * f)(x) \to f(x) \qquad \text{as } \varepsilon \to 0$$

at *every* Lebesgue point of f.

[Recall: x Lebesgue point of f if $\lim_{\varepsilon\to 0} \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} |f(y) - f(x)|dy = 0$.]

Solution 2. Without loss of generality we may assume that $f(0) = 0$, otherwise replace f by $f - f(0)$. We can now estimate that

$$\begin{aligned} |f * \eta_\epsilon(x)| &\le \int_{|x-y|<\epsilon} |f(y)||\eta_\epsilon(x-y)|dx \\ &\le \int_{|x-y|<\epsilon} |f(y)|\epsilon^{-n}|\eta((x-y)/\epsilon)|dx \\ &\le \frac{C||\eta||_\infty}{|B(x,\epsilon)|}\int_{B(x,\epsilon)} |f(y)|dy, \end{aligned}$$

where the last expression converges to zero due to x being a Lebesgue point.

Exercise 3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary $\partial\Omega$ and with trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$. If $\psi \in C^\infty(\overline{\Omega})$, show that

$$\psi T(u) \;=\; T(\psi u), \qquad \text{for all} \quad u \in W^{1,p}(\Omega).$$

Solution 3. Let A_ϕ denote the following operator on $\mathcal{W}^{1,p}(\Omega)$

$$A_\phi u = \phi u$$

and B_ϕ the following operator on $L^p(\partial\Omega)$

$$B_\phi u = \phi u.$$

Then both the operators $A_\phi : \mathcal{W}^{1,p}(\Omega) \to \mathcal{W}^{1,p}(\Omega)$ and $B_\phi : L^p(\partial\Omega) \to L^p(\partial\Omega)$ are bounded. We need to prove that

$$B_\phi T u = T A_\phi u$$

for all $u \in \mathcal{W}^{1,p}(\Omega)$. But this identity is valid in the dense set $C^\infty(\overline{\Omega}) \subset \mathcal{W}^{1,p}(\Omega)$. By the unique linear extension of bounded operators we must have that the identity is valid everywhere, since the operators $A_\phi T$ and $B_\phi T$ are bounded from $\mathcal{W}^{1,p}(\Omega)$ to $L^p(\partial\Omega)$.

Exercise 4. If $u \in W^{1,p}(\Omega)$, show that then $|u| \in W^{1,p}(\Omega)$.

[Hint: Apply Problem 4 in Exercises 2, with the function $f(x) = f_\varepsilon(x) \;=\; \sqrt{x^2+\varepsilon^2} - \varepsilon$, and let $\varepsilon \to 0$.]

Solution 4. As in Exercise 4 in set 2, all we need to do is verify the following equation for the weak derivatives

$$\partial_{x_j} f \circ u = f'(u)\partial_{x_j} u,$$

where $f(x) = |x|$ and $f'(x) = \operatorname{sgn}(x)$, in particular we must define $f'(0) = 0$. Let $f_\epsilon(x) = \sqrt{x^2+\epsilon^2} - \epsilon$. Then by Exercise 4 of set 2 we have

$$\int_\Omega f_\epsilon(u(x))\partial x_j g(x)dx = -\int_\Omega \frac{u(x)}{\sqrt{u(x)^2+\epsilon^2}} \partial x_j u(x) g(x)dx.$$

Note now that

$$\frac{u(x)}{\sqrt{u(x)^2+\epsilon^2}} \to \operatorname{sgn}(u(x))$$

pointwise in x and ϵ as $\epsilon \to 0$. We can then use dominated convergence to take the limit and see that

$$\int_\Omega f(u(x))\partial x_j g(x)dx = -\int_\Omega \operatorname{sgn}(u(x))\partial x_j u(x) g(x)dx.$$

Exercise 5. Show that there are bounded domains $\Omega \subset \mathbb{R}^2$ where the Gagliardo-Nirenberg-Sobolev inequality fails: At least for some $1 \le p < n = 2$ and $p^* = \frac{2p}{2-p}$, there are functions $u \in W^{1,p}(\Omega) \setminus L^{p^*}(\Omega)$.

One possible class of such domains Ω, called "rooms and corridors", is described on the next page.

Rooms and corridors. Let $\Omega \subset \mathbb{R}^2$ be a domain such as in the picture above,

$$\Omega = \bigcup_{k=1}^{\infty} (D_k \cup P_k),$$

where the 'fat' sets D_k, *the rooms*, and the 'thin' sets P_k, *the corridors*, $k = 0, 1, 2, \ldots$, are defined as follows:

Let first $d_k = 1 - 2^{-k}$, $k = 0, 1, 2, \ldots$, and define then the rooms as cubes

$$D_k = (d_{2k}, d_{2k+1}) \times \left(-2^{-2k-2}, 2^{-2k-2}\right)$$

and the corridors as rectangles

$$P_k = [d_{2k+1}, d_{2k+2}] \times \left(-\varepsilon_k 2^{-2k-2}, \varepsilon_k 2^{-2k-2}\right).$$

[Hint for solving Problem 5: Choose u to be constant c_k in each room D_k, and let it grow linearly in each corridor P_k. Choose the constants c_k so that $u \in L^p(\Omega) \setminus L^{p^*}(\Omega)$, and then the thinnesses ε_k suitably to have $u \in W^{1,p}(\Omega)$]

Solution 5. We construct the function and the domain as in the hint, so let our domain Ω be a union of corridors P_k and rooms D_k.

Let us first estimate the integral of $|f(x)|^q$ over Ω for a general exponent $q \ge 1$. If f is the constant c_k in the room D_k, the integral over the room amounts to $c_k^q |D_k|$. The part of the integral corresponding to the corridors will not matter if the corridor P_k has an area less than $|D_k|$, a choice that we will make later. Thus for f to be in L^q we must have that

$$\sum_{k=1}^{\infty} c_k^q |D_k| < \infty.$$

Let us then estimate the L^p-norm of the derivative. On each room D_k the derivatives of f are zero. On the corridors we may define f linearly, in which case the derivative is comparable to the slope of the function. This slope is equal to $(c_{k+1} - c_k)/\ell(P_k)$, where $\ell(P_k)$ denotes the length of the corridor P_k. Thus for the function to be in the Sobolev space $\mathcal{W}^{1,p}(\Omega)$ we must have that

$$\sum_{k=1}^{\infty} \frac{|c_{k+1} - c_k|^p}{\ell(P_k)^p} |P_k| < \infty.$$

Choose $c_k = r_1^k$ for some number $r_1 > 1$. Then $c_{k+1} - c_k$ and c_k are comparable to each other. Let the length of each P_k be r_2^k and their width be r_3^k for some numbers $1 > r_2 > r_3 > 0$. Let the rooms D_k be square with side length r_2^k. Then the Sobolev condition becomes

$$\sum_{k=1}^{\infty}(r_1^p r_2^{1-p} r_3)^k < \infty,$$

in other words $r_1^p r_2^{1-p} r_3 < 1$. Similarly the L^q-condition becomes

$$r_1^q r_2^2 < 1.$$

We can always choose r_3 small enough so that the first condition holds. If we are given any $q \geq 1$, we can choose r_1 and r_2 so that $r_1^q r_2^2 = 1$. In this case $f \in L^{q'}$ only for exponents $q' < q$. Choosing $q \leq p^*$ shows that the Gagliado-Nirenberg-Sobolev inequality can't be true in Ω.

Note that the area of P_k is now less than that of D_k since we could choose $r_3 < r_2$. The domain Ω is also bounded since $r_2 < 1$, which gives that the lengths of the rooms and corridors add to a finite number.