SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 3

Exercise 1. Show that $u(x) = \log\log\left(1+\frac{1}{|x|}\right) \in W^{1,n}(B)$, where $B = B(0,1)$ is the unit ball in $\mathbb{R}^n$.

Solution 1. We calculate first that

$$\partial_{x_j} u(x) = \frac{1}{\log(1+|x|^{-1})} \frac{1}{1+|x|^{-1}} \frac{-1}{|x|^2} \frac{x_j}{|x|}.$$

Thus

$$|\nabla u(x)| = \frac{1}{\log(1+|x|^{-1})} \frac{1}{1+|x|} \frac{1}{|x|}.$$

Since this function is radially symmetric, we can use polar coordinates to compute that

$$\int_{B(0,1)} |\nabla u(x)|^n dx = C_1 \int_0^1 \frac{1}{\log^n(1+r^{-1})} \frac{1}{(1+r)^n} \frac{1}{r^n} r^{n-1} dr \leq C_2 \int_0^1 \frac{1}{\log^n(1+r^{-1})} \frac{1}{r} dr$$

We will show that the right hand side is finite. The only issue is the singularity at $r = 0$. Note that

$$\log^n(1+r^{-1}) \to \infty$$

as $r \to 0$. Thus we might as well lower the exponent $n \geq 2$ and replace $\log(1+r^{-1})$ by $\log(r^{-1}) = -\log(r)$ since both terms are comparable. It is enough to show that

$$\int_0^{\delta_1} \frac{1}{\log^2(r)} \frac{1}{r} dr < \infty$$

for some $\delta_1 > 0$. Make a substitution $r = e^{-t}$. Thus the integral becomes

$$\int_{\delta_2}^{\infty} \frac{1}{t^2} dt.$$

This is finite so we are done. It is also easy to check that

$$\int_{B(0,1)} |u(x)|^n dx < \infty,$$

since the function u has a better singularity at $x = 0$ than its derivatives. This shows that $u \in \mathcal{W}^{1,n}$ for $n \geq 2$.

Exercise 2. a) Show that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$.

b) Prove the generalised Hölder's inequality: If $1 \leq p_1, \ldots p_m \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$, then

$$\int_\Omega |u_1 u_2 \cdots u_m| dx \leq \prod_{j=1}^m \left(\int_\Omega |u_j|^{p_j} dx\right)^{1/p_j}.$$

Solution 2. a) Let $u \in \mathcal{W}^{1,p}(\mathbb{R}^n)$. If u has compact support, then we can approximate it with smooth functions of compact support by Exercise 3 of set 1 - just choose a slightly larger compact subset where you can approximate it. Thus it is enough to show that we can approximate u by Sobolev functions of compact support.

Let φ be a smooth function supported in $B(0,1)$ with $\varphi(0) = 1$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. Define

$$u_\epsilon(x) = \varphi(\epsilon x)u(x).$$

We claim that $u_\epsilon \to u$ in $\mathcal{W}^{1,p}(\mathbb{R}^n)$. It is easy to see that $u_\epsilon \to u$ in $L^p(\mathbb{R}^n)$, so we only check the convergence of the derivatives. For this one can compute that

$$\partial_{x_j}(\varphi(\epsilon x)u(x)) = \epsilon\varphi_{x_j}(\epsilon x)u(x) + \varphi(\epsilon x)u_{x_j}.$$

We can now use dominated convergence to see that the first term goes to zero and the second term goes to u_{x_j} in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0$. This finishes the proof.

b) Due to laziness, we refer the reader to Wikipedia -> Hölder's inequality -> Generalized Hölder's inequality -> proof.

Exercise 3. If $\Omega = (a,b) \subset \mathbb{R}$, how do you define the trace operator T on $W^{1,p}(\Omega)$? If $u \in W^{1,p}(\Omega)$, show that $T(u) = 0$ if and only if the absolutely continuous representative of u satisfies $u(a) = u(b) = 0$.

Solution 3. The trace operator in 1-dimensional space is defined as in Evans' book - one just has to interpret the space $L^p(\partial[a,b])$ correctly. The set $\partial[a,b] = \{a,b\}$ is a measure space with the counting measure, so we can define $L^p(\partial[a,b])$ in the usual sense. In this case $L^p(\{a,b\})$ will be equivalent to $\mathbb{R}^2$ with the norm $|(x,y)| = (|x|^p + |y|^p)^{1/p}$. All norms on $\mathbb{R}^2$ are equivalent though so we may as well use the L^1-norm instead.

Let now $u \in \mathcal{W}^{1,p}([a,b])$. Denote by $\tilde{u}$ its absolutely continuous representative. Then we define an operator

$$\tilde{T} : \mathcal{W}^{1,p}([a,b]) \to L^p(\partial[a,b])$$

by

$$\tilde{T}u = (\tilde{u}(a), \tilde{u}(b)).$$

We first show that this operator is continuous. Let φ_1 be a C^∞-function on $[a,b]$, supported on $[a, a+\epsilon]$ and such that $\varphi_1(a) = 1$. Let similarly φ_2 be supported on $[b-\epsilon, b]$ and $\varphi_2(b) = 1$. Then $\varphi_1\tilde{u}$ and $\varphi_2\tilde{u}$ are also absolutely continuous on $[a,b]$ (not difficult to check directly). Thus

$$\begin{aligned}
|\tilde{u}(a)| &= |\varphi_1(b)\tilde{u}(b) - \varphi_1(a)\tilde{u}(a)| \\
&= \left| \int_a^b (\varphi_1\tilde{u})'(t)dt \right| \\
&\leq \int_a^b |\varphi_1'(t)||\tilde{u}(t)|dt + \int_a^b |\varphi_1(t)||\tilde{u}'(t)|dt \\
&\leq ||\varphi_1'||_q||\tilde{u}||_p + ||\varphi_1||_q||\tilde{u}'||_p \\
&\leq C||\tilde{u}||_{\mathcal{W}^{1,p}([a,b])}.
\end{aligned}$$

Similarly $|\tilde{u}(b)| \leq C||\tilde{u}||_{\mathcal{W}^{1,p}([a,b])}$, so

$$||\tilde{T}(\tilde{u})||_{L^1(\partial[a,b])} = |\tilde{u}(a)| + |\tilde{u}(b)| \leq 2C||u||_{\mathcal{W}^{1,p}([a,b])}.$$

This proves the continuity of $\tilde{T}$.

The conclusion now comes from the fact that $\tilde{T}$ must be the same as the usual trace operator T. This is because clearly $\tilde{T}u = Tu$ when $u \in C^\infty([a,b])$. In the general case we have some approximating sequence of smooth functions u_n of u and

$$\tilde{T}u = \lim_{n\to\infty} \tilde{T}u_n = \lim_{n\to\infty} Tu_n = Tu.$$

This is the unique extension property of continuous linear operators: Every continuous extension of a linear operator from a dense subset is the same. Thus we have found an expression for the trace in the 1-dimensional case: $Tu = (\tilde{u}(a), \tilde{u}(b))$. In particular, $Tu = 0$ iff $\tilde{u}(a) = \tilde{u}(b) = 0$.

Exercise 4. Let $0 \leq \eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier, with $\mathrm{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta(x)dx = 1$. Set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$, and let $e_n := (0,\dots,0,1) \in \mathbb{R}^n$.

If $u \in W^{1,p}(\mathbb{R}^n_+)$ and $u_\varepsilon(x) = u(x + 2\varepsilon e_n)$ show that $w_\varepsilon := \eta_\varepsilon * u_\varepsilon$ is well defined in $\mathbb{R}^n_+$, and

$$w_\varepsilon \in C^\infty(\overline{\mathbb{R}^n_+}) \quad \text{with} \quad \|u - w_\varepsilon\|_{W^{1,p}(\mathbb{R}^n_+)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Solution 4. Recall that if f is in $L^p(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx \to 0$ as $|h| \to 0$. This can be used to find that

$$u_\epsilon(x) = u(x + 2\epsilon e_n) \to u(x)$$

in $\mathcal{W}^{1,p}(\mathbb{R}^n_+)$ as $\epsilon \to 0$. The convolution $w_\epsilon = \eta_\epsilon * u_\epsilon$ is well-defined since

$$w_\epsilon(x) = \int_{\mathbb{R}^n} u(x - y + 2\epsilon e_n)\eta_\epsilon(y)dy,$$

and in the integral we have $\eta_\epsilon(y) = 0$ unless $|y| \leq \epsilon$, in which case $x - y + 2\epsilon e_n \in \mathbb{R}^n_+$ and there is no problem. To see the convergence of w_ϵ to u, we split:

$$u - \eta_\epsilon * u_\epsilon = (u - u_\epsilon) + (u_\epsilon - \eta_\epsilon * u_\epsilon).$$

The term $u - u_\epsilon$ converges to zero in $\mathcal{W}^{1,p}$. For the next term, we have to estimate the L^p-norms of the expressions

$$u_\epsilon - \eta_\epsilon * u_\epsilon \quad \text{and} \quad \partial_{x_j} u_\epsilon - \eta_\epsilon * \partial_{x_j} u_\epsilon.$$

For this we refer to Reaalianalyysi 1, page 31 for the formula

$$||f - f * g||^p_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} |f(x) - f(x-y)|^p dx \right) dy.$$

In the proof one can choose to only integrate over $\mathbb{R}^n_+$ to get:

$$||f - f * g||^p_{L^p(\mathbb{R}^n_+)} \leq \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n_+} |f(x) - f(x-y)|^p dx \right) dy.$$

Let us apply this for $f = u_\epsilon$ and $g = \eta_\epsilon$, giving that:

$$||u_\epsilon - \eta_\epsilon * u_\epsilon||^p_{L^p(\mathbb{R}^n_+)} \leq \int_{\mathbb{R}^n} \eta_\epsilon(y) \left(\int_{\mathbb{R}^n_+} |u(x + 2\epsilon e_n) - u(x - y + 2\epsilon e_n)|^p dx \right) dy.$$

Since η_ϵ is supported in the set $|y| < \epsilon$, we also have $|y| < \epsilon$ in the inner integral. This converges to zero as $\epsilon \to 0$ so we can estimate the inner integral as

$$\int_{\mathbb{R}^n_+} |u(x + 2\epsilon e_n) - u(x - y + 2\epsilon e_n)|^p dx \leq \sup_{|h|<\epsilon} \left\{ \int_{\mathbb{R}^n_+} |u(x) - u(x+h)|^p dx \right\}.$$

The key point is that we can see now that the right hand side goes to zero as $\epsilon \to 0$. This shows that

$$||u_\epsilon - \eta_\epsilon * u_\epsilon||^p_{L^p(\mathbb{R}^n_+)} \to 0.$$

If we put $\partial_{x_j} u_\epsilon$ in place of u_ϵ we also get that

$$||\partial_{x_j} u_\epsilon - \eta_\epsilon * \partial_{x_j} u_\epsilon||^p_{L^p(\mathbb{R}^n_+)} \to 0.$$

Thus

$$||u_\epsilon - w_\epsilon||_{\mathcal{W}^{1,p}(\mathbb{R}^n_+)} \to 0.$$

Exercise 5. Suppose $u \in W^{1,p}(\mathbb{R}^n_+)$ with weak derivatives $D^\alpha u \in L^p(\mathbb{R}^n_+)$, $|\alpha| = 1$, and let $\varphi \in C^1(\overline{\mathbb{R}^n_+})$.

Suppose $\varphi(x) = 0$ if $x \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ or if $|x| > M$. Show that

$$\int_{\mathbb{R}^n_+} u(x)\, D^\alpha \varphi(x)\, dx = -\int_{\mathbb{R}^n_+} D^\alpha u(x)\, \varphi(x)\, dx.$$

[Hint: Use e.g. ideas from proof of Theorem 2/Section 5.5/Evans]

Solution 5. Note that $\varphi \in \mathcal{W}^{1,q}(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$, $\frac{1}{p} + \frac{1}{q} = 1$. Thus we can take the trace of φ and must have that $T\varphi \equiv 0$. By Evans, φ can be approximated by C_0^∞-functions φ_m in $\mathcal{W}^{1,q}(\mathbb{R}^n_+)$. By Hölder's inequality we find that

$$\int_{\mathbb{R}^n_+} u(x) D^\alpha \varphi_m(x) dx \to \int_{\mathbb{R}^n_+} u(x) D^\alpha \varphi(x) dx$$

and

$$-\int_{\mathbb{R}^n_+} D^\alpha u(x) \varphi_m(x) dx \to -\int_{\mathbb{R}^n_+} D^\alpha u(x) \varphi(x) dx.$$

This proves the result.