SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 1

Exercise 1. If dimension $n = 2$, consider the domain $\Omega = B(0,1) \setminus [0,1]$, i.e. the disk minus a slit. Show that in this domain, $C^\infty(\overline{\Omega})$ is *not* dense in $W^{1,p}(\Omega)$.

Solution 1. The philosophical reason why $C^\infty(\overline{\Omega})$ is not dense in $\mathcal{W}^{1,p}(\Omega)$ is the following: If a function is in $C^\infty(\overline{\Omega})$, then it is smooth in the whole closed unit disc $\overline{B(0,1)}$. However, a function in the Sobolev space $\mathcal{W}^{1,p}(\Omega)$ may even have a jump discontinuity on the line segment $[0,1]$ - the limit as x approaches from above and below may not be the same at all. A short way to rigorously prove that $C^\infty(\overline{\Omega})$ is not dense is to check that the following two facts are true:

1. There is a function in the Sobolev space $\mathcal{W}^{1,p}(\Omega)$, but not in $\mathcal{W}^{1,p}(B(0,1))$.
2. If a sequence of $C^\infty(\overline{\Omega})$-functions converges in $\mathcal{W}^{1,p}(\Omega)$, then it also converges in $\mathcal{W}^{1,p}(B(0,1))$.

These two points clearly imply that $C^\infty(\overline{\Omega})$ cannot be dense, since the second point implies that any Sobolev-limit of $C^\infty(\overline{\Omega})$-functions is in $\mathcal{W}^{1,p}(B(0,1))$.

We prove the second fact first. If a function is in $C^\infty(\overline{\Omega}) = C^\infty(\overline{B(0,1)})$, then it is also in the Sobolev space $\mathcal{W}^{1,p}(B(0,1))$. If we have a sequence (u_m) of such functions converging in $\mathcal{W}^{1,p}(\Omega)$, then it is also a Cauchy sequence in $\mathcal{W}^{1,p}(\Omega)$. However, since the segment $[0,1]$ is of zero measure we have that

$$\int_\Omega |u_m - u_n|^p dx = \int_{B(0,1)} |u_m - u_n|^p dx. \int_\Omega |Du_m - Du_n|^p dx \quad = \int_{B(0,1)} |Du_m - Du_n|^p dx.$$

The above two equalities show that (u_m) is also a Cauchy sequence on $\mathcal{W}^{1,p}(B(0,1))$. Note also a very important point: The weak derivative of each u_m as defined on the set Ω is the same as the weak derivative on the set $B(0,1)$. This fact is almost trivial but it is essential to notice that the weak derivatives of u_m are actually well defined on $B(0,1)$. This fact does not hold for general Sobolev functions as we shall soon see.

Let us now prove the point number 1. To construct such a function, it is enough to find a function in $\mathcal{W}^{1,p}(\Omega)$ which does not have weak derivatives on the disc $B(0,1)$. Take a compactly contained subinterval $[a,b] \subset [0,1]$. Let f be a smooth function on the closed upper half plane which is equal to the constant 1 on $[a,b]$ and in a small neighbourhood of this interval. Let f be such that it is positive on a slightly larger neighbourhood of $[a,b]$ but zero everywhere else. Extend f as zero to the lower half plane. Then one can check that f is in $\mathcal{W}^{1,p}(\Omega)$ but has a jump discontinuity on the interval $[a,b]$. This implies that f does not have a weak derivative, essentially by the same arguments as in Exercise 1 of Set 1. Thus we are done.

Exercise 2. Integrate by parts and approximate, to prove the interpolation inequality

$$\int_\Omega |Du|^2 dx \leq C \left(\int_\Omega u^2 dx\right)^{1/2} \left(\int_\Omega |D^2u|^2 dx\right)^{1/2}$$

for all $u \in W_0^2(\Omega)$, the closure of $C_c^\infty(\Omega)$ in $W^2(\Omega)$.

Solution 2. Let first $\Omega \in C_0^\infty(\Omega)$. Integration by parts gives the identity

$$\int_\Omega |Du|^2 dx = \int_\Omega \sum_{j=1}^n u_{x_j} u_{x_j} dx = -\int_\Omega \sum_{j=1}^n u\, u_{x_j x_j} dx.$$

We can now estimate the right hand side with Cauchy-Schwartz:

$$-\int_\Omega \sum_{j=1}^n u\, u_{x_j x_j} dx \leq \left(\int_\Omega |u|^2 dx\right)^{1/2} \left(\int_\Omega \left|\sum_{j=1}^n u_{x_j x_j}\right|^2 dx\right)^{1/2}.$$

We now use the following basic inequality: There is a constant $C_n > 0$ such that

$$(a_1 + a_2 + \cdots + a_n)^2 \leq C_n \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)$$

for any positive real numbers $a_1, \ldots, a_n$. Applying this gives that

$$\int_\Omega \left|\sum_{j=1}^n u_{x_j x_j}\right|^2 dx \leq C_n \int_\Omega \sum_{j=1}^n \left|u_{x_j x_j}\right|^2 dx \leq C_n \int_\Omega |D^2u|^2 dx.$$

Combining everything gives the desired inequality

$$\int_\Omega |Du|^2 dx \leq \sqrt{C_n} \left(\int_\Omega |u|^2 dx\right)^{1/2} \left(\int_\Omega |D^2u|^2 dx\right)^{1/2}. \tag{1}$$

Take now any $u \in \mathcal{W}_0^{2,2}(\Omega)$, and approximate u by C_0^∞-functions u_m in the Sobolev norm. Thus we find that

$$\int_\Omega |Du_m|^2 dx \to \int_\Omega |Du|^2 dx$$
$$\int_\Omega |u_m|^2 dx \to \int_\Omega |u|^2 dx$$
$$\int_\Omega |D^2u_m|^2 dx \to \int_\Omega |D^2u|^2 dx$$

as $m \to \infty$. Since inequality (1) holds for each u_m, it must then also hold for u. Thus we are done.

Exercise 3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, and assume that every point $x_0 \in \partial\Omega$ has a neighbourhood V such that $C^\infty(\overline{\Omega \cap V})$ is dense in $W^{1,p}(\Omega \cap V)$. Show that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Solution 3. Since Ω is bounded its boundary is compact. Choose a finite cover $V_1, \ldots, V_N$ of $\partial\Omega$ such that $C^\infty(\overline{\Omega \cap V_j})$ is dense in $\mathcal{W}^{1,p}(\Omega \cap V_j)$ for each j. Choose also a compactly contained open set $V_0 \subset\subset \Omega$ such that $V_0, V_1, \ldots, V_N$ is a cover of the whole set Ω. Let $\psi_0, \ldots, \psi_N$ be a partition of unity on the sets V_j.

Take now any $u \in \mathcal{W}^{1,p}(\Omega)$. For each j, choose a sequence $u_m^{(j)} \in C^\infty(\overline{\Omega \cap V_j})$ converging to $u|_{V_j}$ in $\mathcal{W}^{1,p}(\Omega \cap V_j)$ (possible also for $j = 0$ due to exercise 3 of set 1). Define

$$u_m = \sum_{j=0}^{N} u_m^{(j)} \psi_j.$$

Then $u_m \in C^\infty(\overline{\Omega})$. Moreover,

$$||u - u_m||_{\mathcal{W}^{1,p}(\Omega)} \leq \sum_{j=0}^{N} \left\| \psi_j (u - u_m^{(j)}) \right\|_{\mathcal{W}^{1,p}(\Omega \cap V_j)}.$$

The right hand side goes to zero as $m \to \infty$, which shows that $C^\infty(\overline{\Omega})$ is dense in $\mathcal{W}^{1,p}(\Omega)$.

Exercise 4. Suppose $f \in C^1(\mathbb{R})$ with $f' \in L^\infty$ and $f(0) = 0$. If $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, show that $f \circ u \in W^{1,p}(\Omega)$ and we have the chain rule

$$D^\alpha (f \circ u)(x) = f'(u) D^\alpha u(x), \qquad |\alpha| = 1,$$

almost everywhere in Ω.

[Hint: Approximation can be useful also in this problem]

Solution 4. We first prove that the formula

$$\partial_{x_j}(f \circ u) = f'(u) \partial_{x_j} u \tag{2}$$

holds for Sobolev functions u, where both of the derivatives are interpreted as the weak derivatives. When u is in $C_0^\infty(\Omega)$, we have the pointwise formula $\partial_{x_j}(f \circ u)(x) = f'(u(x))\partial_{x_j} u(x)$ and thus we also have the weak version of the formula,

$$\int_\Omega (f \circ u) \partial_{x_j} \phi \, dx = -\int_\Omega f'(u)(\partial_{x_j} u) \phi \, dx,$$

for any test function ϕ. Let us now try to prove that this formula holds for general $u \in \mathcal{W}^{1,p}(\Omega)$ by approximation. Let ϕ be fixed, and approximate u by smooth compactly supported functions u_m in $\mathcal{W}^{1,p}(\operatorname{supp} \phi)$. We need to prove that

$$\int_\Omega (f \circ u_m) \partial_{x_j} \phi \, dx \to \int_\Omega (f \circ u) \partial_{x_j} \phi \, dx$$
$$\int_\Omega f'(u_m)(\partial_{x_j} u_m) \phi \, dx \to \int_\Omega f'(u)(\partial_{x_j} u) \phi \, dx$$

as $m \to \infty$. For the first limit, we estimate by the mean value theorem and Hölder's inequality that

$$\begin{aligned}\left|\int_\Omega (f\circ u)\partial_{x_j}\phi\,dx - \int_\Omega (f\circ u_m)\partial_{x_j}\phi\,dx\right| &\le \int_\Omega |(f\circ u)-(f\circ u_m)||\partial_{x_j}\phi|\,dx\\ &\le \int_\Omega ||f'||_\infty |u-u_m||\partial_{x_j}\phi|\,dx\\ &\le ||f'||_\infty \left(\int_{\operatorname{supp}\phi} |u-u_m|^p\right)^{1/p}\left(\int_{\operatorname{supp}\phi}|\partial_{x_j}\phi|^q\right)^{1/q}.\end{aligned}$$

The fact that $u_m \to u$ in L^p gives that the right hand side converges to zero.

For the second limit, we estimate that

$$\begin{aligned}&\left|\int_\Omega f'(u)(\partial_{x_j}u)\phi\,dx - \int_\Omega f'(u_m)(\partial_{x_j}u_m)\phi\,dx\right|\\ &\quad\le \int_\Omega |f'(u)-f'(u_m)||\partial_{x_j}u||\phi|\,dx + \int_\Omega |f'(u_m)||\partial_{x_j}u-\partial_{x_j}u_m||\phi|\,dx\end{aligned}$$

The terms on the right hand side can be dealt with as follows. For the first term it is enough to use dominated convergence. We may assume without loss of generality that $u_m \to u$ pointwise almost everywhere. This can be justified by two reasons: The first reason is that u_m may be defined via mollifiers as in the last exercise set, and pointwise a.e. convergence holds for the mollified sequence (as proven in the course Reaalianalyysi I). The second reason is that a L^p converging sequence always has a pointwise a.e. converging subsequence, again by Reaalianalyysi I.

The fact that f' is continuous and that $u_m \to u$ pointwise a.e. implies that

$$|f'(u)-f'(u_m)||\partial_{x_j}u||\phi| \to 0$$

pointwise a.e. Since we also have the bound

$$|f'(u)-f'(u_m)||\partial_{x_j}u||\phi| \le 2||f'||_\infty|\partial_{x_j}u||\phi|$$

and the right hand side is in $L^1(\Omega)$, dominated convergence gives that

$$\int_\Omega |f'(u)-f'(u_m)||\partial_{x_j}u||\phi|\,dx \to 0.$$

For our second term we estimate by Hölder's inequality that

$$\begin{aligned}\int_\Omega |f'(u_m)||\partial_{x_j}u-\partial_{x_j}u_m||\phi|\,dx &\le ||f'||_\infty||\phi||_\infty \int_{\operatorname{supp}\phi}|\partial_{x_j}u-\partial_{x_j}u_m|\,dx\\ &\le ||f'||_\infty||\phi||_\infty|\operatorname{supp}\phi|^{1-1/p}\left(\int_{\operatorname{supp}\phi}|\partial_{x_j}u-\partial_{x_j}u_m|^p\,dx\right)^{1/p}.\end{aligned}$$

The right hand side converges to zero so we are done. This proves our formula (2), and from this it is easy to deduce that $f\circ u \in \mathcal{W}^{1,p}(\Omega)$.

Exercise 5. Suppose $\Phi : U \to V$ is a C^1-diffeomorphism between domains $U, V \subset \mathbb{R}^n$; in particular, $\Phi^{-1} : V \to U$ is also a C^1-smooth homeomorphism.

If $\Omega \subset\subset U$ and $\Omega' = \Phi(\Omega)$, show that

$$u \in W^{1,p}(\Omega') \Leftrightarrow \; u \circ \Phi \in W^{1,p}(\Omega),$$

with $c_1\|u\|_{W^{1,p}(\Omega')} \leq \|u \circ \Phi\|_{W^{1,p}(\Omega)} \leq c_2\|u\|_{W^{1,p}(\Omega')}$.

Solution 5. Suppose first that $u(y_1, \dots, y_n)$ is in $C_0^\infty(\Omega')$. Then we have the formula

$$\partial_{x_j}(u \circ \Phi) = \sum_{k=1}^{n} (u_{y_k} \circ \Phi)\, \Phi_{x_j}^{(k)}, \tag{3}$$

where $\Phi = (\Phi^{(1)}, \dots, \Phi^{(n)})$ is our C^1-diffeomorphism. The weak version of this formula is

$$\int_\Omega (u \circ \Phi)\partial_{x_j} g \, dx = \int_\Omega g \sum_{k=1}^{n} (u_{y_k} \circ \Phi)\, \Phi_{x_j}^{(k)} \, dx, \tag{4}$$

where g is a test function. Let $\Psi = \Phi^{-1}$. Making a change of variables $x \mapsto \Psi(y)$ gives

$$\int_{\Omega'} u(y)(\partial_{x_j} g \circ \Psi(y))|J_\Psi(y)| \, dy = \int_{\Omega'} (g \circ \Psi)(y) \left(\sum_{k=1}^{n} u_{y_k}(y) \left(\Phi_{x_j}^{(k)} \circ \Psi(y) \right) \right) |J_\Psi(y)| \, dy.$$

These integrals are still over compact sets, so we may use approximation to prove that this formula also holds when u is in the Sobolev class $\mathcal{W}^{1,p}(\Omega')$. The formula may look daunting but it can be split into terms of the form

$$\int_{\Omega'} u(y) G_1(y) dy \quad \text{and} \quad \int_{\Omega'} u_{x_j}(y) G_2(y) dy,$$

where G_1 and G_2 are continuous functions. Approximating such terms is a simple application of Hölder's inequality as in Exercise 4.

We may now apply the change of variables formula (valid also when u and its derivatives are only measurable) backwards to conclude that identity (4) also holds when u is in $\mathcal{W}^{1,p}(\Omega')$. Thus formula (3) holds for Sobolev functions when the derivatives are interpreted in the weak sense. We also have the estimate

$$\begin{aligned}
\int_\Omega |\partial_{x_j}(u \circ \Phi)|^p dx &= \int_\Omega \left| \sum_{k=1}^{n} (u_{y_k} \circ \Phi)\, \Phi_{x_j}^{(k)} \right|^p dx \\
&\leq C_1 \int_\Omega \sum_{k=1}^{n} |\, (u_{y_k} \circ \Phi)\, |^p |\Phi_{x_j}^{(k)}|^p dx \\
&= C_1 \int_{\Omega'} \sum_{k=1}^{n} |u_{y_k}|^p |\Phi_{x_j}^{(k)} \circ \Psi|^p |J_\Psi| dy \\
&\leq C_2 \sum_{k=1}^{n} \int_{\Omega'} |u_{y_k}|^p dy.
\end{aligned}$$

The last estimate comes from the fact that derivatives of Φ and Ψ are bounded on compact subsets (such as Ω and Ω'). Similarly

$$\int_{\Omega} |(u \circ \Phi)|^p dx \leq C_3 \int_{\Omega'} |u|^p dy.$$

Combining these we obtain that

$$||u \circ \Phi||_{\mathcal{W}^{1,p}(\Omega)} \leq C||u||_{\mathcal{W}^{1,p}(\Omega')}.$$

Since the same estimates hold for the inverse Ψ, we also have that

$$c||u||_{\mathcal{W}^{1,p}(\Omega')} \leq ||u \circ \Phi||_{\mathcal{W}^{1,p}(\Omega)}.$$

This concludes the proof.