SOBOLEV SPACES. (spring 2016)

MODEL SOLUTIONS FOR SET 1

Exercise 1. If $u(x) = |x|,\ x \in (-1, 1)$, show that u does not have the weak second order derivative u'' in $\Omega = (-1, 1)$.

[Recall that as discussed in the lectures, the weak first order derivative exists, $u'(x) = \operatorname{sign}(x)$.]

Solution 1. From the lectures we remember that $u'(x) = \operatorname{sign}(x)$ in the sense of weak derivatives. It remains to show that $\operatorname{sign}(x)$ does not have a locally integrable weak derivative. If this were not the case there would be $v \in L^1_{loc}$ such that

$$\int_{\mathbb{R}} v(x)g(x)dx = -\int_{\mathbb{R}} \operatorname{sign}(x)g'(x)dx$$

for every test function g. We compute that

$$-\int_{\mathbb{R}} \operatorname{sign}(x)g'(x)dx = -\int_0^{\infty} g'(x)dx - \int_{-\infty}^0 g'(x)dx = -2g(0)$$

Thus

$$\int_{\mathbb{R}} v(x)g(x)dx = -2g(0)$$

for all test functions g. But we can choose a uniformly bounded sequence (g_n) of test functions such that $g_n(0) = 1$ and $\lim_{n\to\infty} g_n(x) = 0$ for $x \neq 0$. Dominated convergence gives that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}} v(x)g_n(x)dx = \lim_{n\to\infty} -2g_n(0) = -2,$$

a contradiction.

Exercise 2. If $u \in W^{k,p}(\Omega)$ and $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, set

$$u^\varepsilon(x) = (\eta_\varepsilon * u)(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y - x)u(y)dy, \qquad x \in \Omega_\varepsilon.$$

Show that $u^\varepsilon(x)$ is a well defined C^∞-function in Ω_ε and that

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \qquad \text{pointwise in } \Omega_\varepsilon.$$

Solution 2. The smoothness of $\eta_\epsilon * u$ was proven in the course Reaalianalyysi 1 (see Lause 2.26). It also included the formula

$$\partial_{x_j}\left(\eta_\epsilon * u\right) = \left(\partial_{y_j}\eta_\epsilon\right) * u,$$

which essentially comes from differentiation under the integral sign. We now use the definition of weak derivatives to see that

$$\left(\partial_{y_j}\eta_\epsilon\right) * u = \int_{\mathbb{R}^n} \partial_{y_j}\eta_\epsilon(x-y)u(y) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)\partial_{y_j}u(y) = \eta_\epsilon * \left(\partial_{y_j}u\right).$$

By induction we find that $D^\alpha u_\epsilon = \eta_\epsilon * D^\alpha u$.

Exercise 3. If u_ε and Ω_ε are as in the previous problem, show that $u_\varepsilon \in W^{k,p}(\Omega_\varepsilon)$ and that $u^\varepsilon \to u$ in $W^{k,p}_{loc}(\Omega)$. That is: if $U \subset \Omega$ is an open subset with compact closure $\overline{U} \subset \Omega$, then $u^\varepsilon \to u$ in $W^{k,p}(U)$ as $\varepsilon \to 0$.

Solution 3. Let $U \subset \Omega$ be a fixed open subset with compact closure. Then for sufficiently small ϵ we have $U \subset \Omega_\epsilon$, which makes the functions u_ϵ well-defined on U. By the previous exercise we have

$$||D^\alpha u - D^\alpha u_\epsilon||_{L^p(U)} = ||D^\alpha u - \eta_\epsilon * D^\alpha u||_{L^p(U)}.$$

We now appeal to the course Reaalianalyysi 1, where it was proven that

$$\lim_{\epsilon\to 0} ||f - \eta_\epsilon * f||_{L^p(U)} = 0$$

for all $f \in L^p(U)$. Applying this result for $f = D^\alpha u$ with $|\alpha| \le k$ gives what we wanted.

Exercise 4. If $u \in W^{1,p}(\Omega)$ and the weak derivative $Du = 0$, show that $u(x) = C$ for a.e. x, for some constant C.

Solution 4. Let $Du = 0$ in the weak sense. Let η_ϵ be a sequence of mollifiers, and define $u_\epsilon = \eta_\epsilon * u$. Then by Exercise 2 we have

$$\partial_{x_j} u_\epsilon = \eta_\epsilon * \partial_{x_j} u = 0.$$

Thus $Du_\epsilon = 0$. Since u_ϵ is smooth, it must be a constant function $u_\epsilon = c_\epsilon$. We know that $u_\epsilon \to u$ in $\mathcal{W}^{1,p}(U)$ for each compact $U \subset \Omega$. We may now choose a subsequence u_{ϵ_n} that converges pointwise almost everywhere (any L^p-converging sequence has such a subsequence). Thus we have $u(x) = \lim_{n\to\infty} c_{\epsilon_n}$ almost everywhere, showing that u is constant.

Exercise 5.[Evans, problem 5.10.6] If $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, show that

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, dt\right)^{1/p} \qquad \text{for } a.e. \ x, y \in [0,1].$$

[Hint: Recall that in dimension $n = 1$ we have a characterisation of $W^{1,p}(0,1)$ in terms of absolutely continuous functions]

Solution 5. By the lectures, we know that u is absolutely continuous. This lets us write

$$u(x) - u(y) = \int_y^x u'(t)dt.$$

We now use Hölder's inequality to estimate that

$$|u(x)-u(y)| \leq \int_y^x 1\cdot|u'|dt \leq \left(\int_y^x 1dt\right)^{(p-1)/p} \left(\int_y^x |u'|^p dt\right)^{1/p} \leq |x-y|^{1-1/p} \left(\int_0^1 |u'|^p dt\right)^{1/p}.$$