Some additional details / material covered in the lectures

1. If $n=1$, every $u \in W^{1,p}(\Omega)$ has abs. cont. representative.

2. Higher order reflection.

3. Partitions of unity.

4. Gagliardo – Nirenberg – Sobolev inequality

5. Morrey's inequality

6. On compact embeddings

7. Remark on compact embeddings of $W^{1,p}(\Omega)$

8. On weak convergence and weak compactness

9. A lemma on weak limits in $L^p(\Omega)$

10. Finding weak solutions via Riesz representation thm.

11. On Dirichlet's Principle.

12. Weierstrass example where no minimizer exists.

13. NEXT PAGE

(13) Further results from Functional Analysis and weak convergence

(14) Return to Dirichlet Principle

Proposition: If $n=1$, $1 \le p < \infty$ and $-\infty < a < b < \infty$ then $u \in W^{1,p}(a,b) \iff u(x) = f(x)$ for a.e. $x \in (a,b)$, where f is absolutely continuous and $f' \in L^p(a,b)$

In particular, $u(x) - u(y) = \int_y^x u'(t)\,dt$ for a.e. $x, y \in [a,b]$

Remark: above f' is the pointwise derivative that exists for absolutely continuous functions at a.e. $x \in (a,b)$.

Proof: "$\Rightarrow$" If Du denotes the weak derivative of u, let $(*)$ $\quad f(x) := \int_a^x Du(t)\,dt$

Since $Du \in L^p(a,b) \subset L^1(a,b)$, f is absolutely continuous [see Real Analysis I, notes p. 58] and so $f'(x)$ exist for a.e x, [see — " — — " —].

Claim: f' is also the weak derivative of f.

Indeed, if $\varphi \in C_c^\infty(a,b) \Rightarrow \varphi f$ is abs. continuous, and so [by Real Analysis I, p. 58] $\Rightarrow$

$$0 = (\varphi f)(b) - (\varphi f)(a) = \int_a^b (\varphi f)'(t)\,dt = \int_a^b \varphi'(t) f(t)\,dt + \int_a^b \varphi(t) f'(t)\,dt$$

$$\therefore \int_a^b f \varphi'\,dt = -\int_a^b f'(t)\varphi(t)\,dt \Rightarrow f' \text{ the weak derivative}$$

But $(*)$ and Real Anal. I, p. 58 $\Rightarrow f'(x) = Du(x)$ for a.e. x

Thus the weak derivative of $u-f$, $D(u-f) = 0$, and $u-f$ is constant a.e. (Exercises 1) Finally $u = f + C$ is abs. continuous. "$\Leftarrow$" clear by above $\square$

Lemma (higher order reflection). Let $B = B(0,1) \subset \mathbb{R}^n$ and $B_+ = B \cap \{x_n \geq 0\}$. If $u \in C^\infty(B_+)$, define its extension to B,

$$\bar{u}(x) = \begin{cases} u(x), & x \in B_+ \\ \sum_{j=1}^{k+1} \alpha_j u(x_1, \dots, x_{n-1}, -\beta_j x_n), & x \in B \setminus B_+ \end{cases}.$$

If $\beta_j > 0 \ \forall j$ and

$$\alpha_1 \beta_1^l + \alpha_2 \beta_2^l + \dots + \alpha_{k+1} \beta_{k+1}^l = (-1)^l, \ l = 0, \dots, k,$$

then

$$u \in C^k(B).$$

Example: Evans, (3)/p. 255, takes $k=1$, $\alpha_1 = -3$, $\alpha_2 = 4$, $\beta_1 = 1$, $\beta_2 = 1/2$

Proof of lemma: Only continuity of $\bar{u}$ and $D^\alpha \bar{u}$ on $\{x_n = 0\}$ needs to be checked.

1° $\lim_{x_n \to 0^+} \bar{u}(x_1, \dots, x_{n-1}, x_n) = \lim_{x_n \to 0^-} \bar{u}(x_1, \dots, x_{n-1}, x_n)$

$\iff \alpha_1 + \alpha_2 + \dots + \alpha_{k+1} = 1$

2° $\lim_{x_n \to 0^+} \partial_{x_n} \bar{u}(x_1, \dots, x_{n-1}, x_n) = \lim_{x \to 0^-} \partial_{x_n} \bar{u}(x_1, \dots, x_{n-1}, x_n)$

$\iff \alpha_1(-\beta_1) + \alpha_2(-\beta_2) + \dots + \alpha_{k+1}(-\beta_{k+1}) = 1$

as seen by derivating the expressions of $\bar{u}$.

3° Continuity of $\partial_{x_n}^l \bar{u} \iff \alpha_1(-\beta_1)^l + \dots + \alpha_{k+1}(-\beta_{k+1})^l = 1$ follows similarly. Same holds for $D^\alpha \bar{u} = \partial_{x_n}^{\alpha_n} \left(\partial_{x_1}^{\alpha_1} \dots \partial_{x_{n-1}}^{\alpha_{n-1}} \bar{u} \right)$.

$\square$

Remarks (a) Since extension $\bar{u}$ is obtained sums of linear coordinate changes, we have

$$(\ast) \qquad \|\bar{u}\|_{W^{k,p}} \leq C_{k,p} \|u\|_{W^{k,p}(B_+)}$$

b) $u \longmapsto \bar{u}$ is linear

c) With above higher order reflection the Extension theorem (5.4. Theorem 1) from Evans generalizes to $W^{k,p}$-spaces, with same proof:

Theorem Suppose $\Omega \subset \mathbb{R}^n$ is bounded domain with C^k-boundary [Evans; Appendix C.1]. Suppose $\Omega \subset\subset V \subset \mathbb{R}^n$. Then there is a bounded linear operator, called extension operator,

$$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n),$$

such that for each $u \in W^{k,p}(\Omega)$ we have

(i) $Eu = u$ a.e. in Ω

(ii) $\operatorname{supp}(Eu) \subset V$

(iii) $\|Eu\|_{W^{k,p}(\Omega)} \leq C \|u\|_{W^{k,p}(\mathbb{R}^n)}$

where the constant C depends only on k, p, Ω and V.

Partition of unity

Lemma If $V \subset\subset \Omega$ [i.e. $\overline{V} \subset \Omega$ compact], then $\exists\ \phi \in C_c^\infty(\Omega)$ with $\phi(x) = 1\ \forall x \in V$, $0 \leq \phi \leq 1\ \forall x \in \Omega$.

Proof: If $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$ mollifier $\left[0 \leq \eta \in C_c^\infty(\mathbb{R}^n) \right.$ with $\int_{\mathbb{R}^n} \eta\, dx = 1$ and $\left. \operatorname{supp}(\eta) \subset B(0,1) \right]$

write $V^\delta := \{ x \in \mathbb{R}^n : \operatorname{dist}(x, V) < \delta \}$ and choose $\delta > 0$ so small that $V^{2\delta} \subset\subset \Omega$.

If $0 < \varepsilon < \delta$, set $\phi(x) = (\eta_\varepsilon * \chi_{V^\delta})(x)$

Since $\operatorname{supp}(\phi) \subset V^{\delta+\varepsilon} \subset\subset \Omega$, $\phi \in C_c^\infty(\Omega)$.

Also $0 \leq \phi(x) = \int_{V^\delta} \eta_\varepsilon(x-y)\, dy \leq \int_{\mathbb{R}^n} \eta_\varepsilon\, dx = 1$

and $x \in V \Rightarrow \eta_\varepsilon(x-y) = 0$ whenever $\operatorname{dist}(y, V) \geq \delta > \varepsilon$.

Thus $\phi(x) = \int_{\mathbb{R}^n} \eta_\varepsilon\, dy = 1\ \forall x \in V$. $\square$

Proposition If $\Omega \subset\subset \bigcup_{j=1}^N V_j$ bounded, then can find

$$\phi_j \in C_c^\infty(V_j), \quad 0 \leq \phi_j \leq 1 \qquad \forall j = 1, \dots, N,$$

such that
$$\sum_{j=1}^N \phi_j(x) = 1 \qquad \forall x \in \Omega.$$

[Call $\{\phi_j\}_1^N$ a partition of unity in Ω, subordinate to covering $\{V_j\}$.]

Proof of Proposition :

Choose first $\phi \in C_c^\infty\left(\bigcup_{j=1}^{N} V_j\right)$ such that

$$0 \le \phi \le 1 \quad \text{and} \quad \phi(x) = 1 \quad \text{for } x \in \overline{\Omega} .$$

(use Lemma on previous page, p. 3a)

Next choose compact sets $K_j \subset V_j$, $j = 1, \dots, N$, such that $\operatorname{supp}(\phi) \subset \bigcup_{j=1}^{N} K_j$.

[every point $x \in \operatorname{supp}(\phi)$ has closed neighbourhoods contained in some V_j ; by compactness can cover $\operatorname{supp}(\phi)$ by finitely many interiors of such neighbourhoods; collect to K_j those neighbourhoods that are contained in V_j]

Now find $\psi_j \in C_c^\infty(V_j)$ with $0 \le \psi_j \le 1$ and $\psi_j\big|_{K_j} = 1$ (by Lemma) $(j = 1, \dots, N)$

Finally, set

$$\phi_1 = \phi\psi_1 , \quad \phi_2 = \phi\psi_2(1-\psi_1) , \quad \dots , \quad \phi_N = \phi\psi_N(1-\psi_1)\cdots(1-\psi_{N-1})$$

Then $0 \le \phi_j \le 1$, $\phi_j \in C_c^\infty(V_j)$ and

$$\sum_{j=1}^{N} \phi_j - \phi \equiv -\phi \prod_{j=1}^{N}(1-\psi_j) = 0 , \quad x \in \mathbb{R}^n$$

since at every point either $\phi = 0$ or some $(1-\psi_j) = 0$.

As $\phi|_\Omega$, we are done . $\square$

Below is a version Gagliardo–Nirenberg–Sobolev inequality, slightly different from Evans Theorem 1 / section 5.6.1

Theorem (Gagliardo–Nirenberg–Sobolev inequality)

Let $1 \leq p < n$ and $p^* = \frac{np}{n-p}$. Then for all $u \in W^{1,p}(\mathbb{R}^n)$ we have

$$(*) \qquad \| u \|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p) \, \| Du \|_{L^p(\mathbb{R}^n)}$$

Proof: Evans proves $(*)$ for $u \in C^1_c(\mathbb{R}^n)$ on pp. 263–265.

Thus we need to extend $(*)$ from $C^1_c(\mathbb{R}^n)$ to all $u \in W^{1,p}(\mathbb{R}^n)$.

For this, if $u \in W^{1,p}(\mathbb{R}^n) \Rightarrow u \in W^{1,p}_0(\mathbb{R}^n)$ [Exercises 3]

Thus have $u_j \in C^\infty_c(\mathbb{R}^n)$ with $\| u_j - u \|_{W^{1,p}(\mathbb{R}^n)} \to 0$

In particular $\| u_j - u \|_{L^p(\mathbb{R}^n)} \underset{(j \to \infty)}{\longrightarrow} 0 \quad \Rightarrow$

For a subsequence,

$$u_{j_\ell}(x) \longrightarrow u(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

[Real Analysis I Lause 1.4.3]

But $\{ u_{j_\ell} \}_{\ell=1}^\infty$ is Cauchy in $L^{p^*}(\mathbb{R}^n)$:

$(*) \Rightarrow$

$$\| u_{j_\ell} - u_{j_m} \|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p) \, \| Du_{j_\ell} - Du_{j_m} \|_{L^p(\mathbb{R}^n)} \underset{(\ell, m \to \infty)}{\longrightarrow} 0$$

Therefore have $v \in L^{p^*}(\mathbb{R}^n)$ s.t. $u_{j_\ell} \to v$ in $L^{p^*}(\mathbb{R}^n)$

$\Rightarrow$ again, have subsequence $u_{j_{\ell_k}}(x) \longrightarrow v(x)$ for a.e. $x \in \mathbb{R}^n$.

Thus $v = u$ a.e. and so $u \in L^{p^*}(\mathbb{R}^n)$!

But now, $\|u\|_{L^{p^*}} \le \|u-u_{j_\ell}\|_{L^{p^*}(\mathbb{R}^n)} + \|u_{j_\ell}\|_{L^{p^*}(\mathbb{R}^n)} \overset{(*)}{\le}$

$$\varepsilon_{j_\ell} + C(n,p)\,\|Du_{j_\ell}\|_{L^p(\mathbb{R}^n)} \xrightarrow[(\ell\to\infty)]{} C(n,p)\,\|Du\|_{L^p(\mathbb{R}^n)}$$

$\square$

Corollaries

Corollary (Sobolev embedding) $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$.

Corollary (Evans Theorem 2 / Section 5.6.1) If $\Omega \subset \mathbb{R}^n$ bdd domain with C^1-boundary and $1 \le p < n$, then

$$\|u\|_{L^{p^*}(\Omega)} \le C\,\|u\|_{W^{1,p}(\Omega)}, \qquad \forall u \in W^{1,p}(\Omega).$$

Proof: Ω admits extension operator $E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ (Evans Section 5.4). Thus $\|u\|_{L^{p^*}(\Omega)} \le \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \overset{\text{p. 4a}}{\le} C\,\|DEu\|_{L^p}$

$$\le C\,\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \overset{\text{Sect. 5.4}}{\le} C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad \square$$

Corollary (Evans Theorem 3 / Section 5.6.1)

If $\Omega \subset \mathbb{R}^n$ any bdd domain and $u \in W_0^{1,p}(\Omega)$, $1 \le p < n$

then $$\|u\|_{L^q(\Omega)} \le C\,\|Du\|_{L^p(\Omega)}, \quad 1 \le q \le p^*$$

Proof: $\left.\begin{array}{l}\text{Hölder}\\ \Omega \text{ bdd}\end{array}\right\} \Rightarrow \|u\|_{L^q(\Omega)} \le C\,\|u\|_{L^{p^*}(\Omega)}$. Since $C_c^\infty(\Omega)$ dense in $W_0^{1,p}(\Omega)$, same proof as for Theorem/p. 4a gives the claim.

$\square$

Morrey's inequality

Here is a little different approach to Morrey's inequality, Theorems 4 & 5 in Evans, Section 5.6.2.

Theorem (Morrey's inequality) If $u \in W^{1,p}(\mathbb{R}^n)$ where $n < p < \infty$, then

$$|u(x) - u(y)| \le C(n,p)\, |x-y|^{1-n/p} \, \|Du\|_{L^p(\mathbb{R}^n)}$$

for a.e. $x, y \in \mathbb{R}^n$.

Note: Since in general a Sobolev function is defined only almost everywhere, that is best one can say in above inequality! But the result implies that for $p > n$, the Sobolev fcn has a (Hölder-) continuous representative!!

Proof of Theorem: 1° Assume $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.

If $x, y \in B(x_0, r)$,

$\boxed{Du = \nabla u}$ recall

$$u(x) - u(y) = \int_0^1 Du(tx + (1-t)y) \cdot (x-y)\, dt$$

in Evans

Write: $$u_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(x)\,dx = \fint_{B(x_0,r)} u(x)\,dx$$

Thus

$$\left| u_{B(x_0,r)} - u(y) \right| = \left| \fint_{B(x_0,r)} \int_0^1 Du(tx + (1-t)y) \cdot (x-y)\, dt\, dx \right|$$

$$\underset{\text{Fubini}}{\leq} \frac{C}{r^n}\int_0^1\int_{B(x_0,r)} |Du(tx+(1-t)y)|\,\underbrace{|x-y|}_{\leq r}\,dx\,dt$$

$$\leq \frac{C}{r^{n-1}}\int_0^1 \frac{1}{t^n}\int_{B(tx_0+(1-t)y,\,tr)} |Du(z)|\,dz\,dt$$

$z = tx+(1-t)y$
$dx = dz/t^n$

$$\leq \frac{C}{r^{n-1}}\int_0^1 \frac{1}{t^n}\left[\int_{B(tx_0+(1-t)y,\,tr)} |Du(z)|^p dz\right]^{1/p} \underbrace{|B(tx_0+(1-t)y,tr)|^{\frac{1}{q}}}_{=c_n(tr)^{n/q}}\,dt$$

Hölder
$\frac{1}{p}+\frac{1}{q}=1$

Since $B(tx_0+(1-t)y, tr) \subset B(x_0,r)$ [check!], we have

$$|u_{B(x_0,r)} - u(y)| \leq C\frac{r^{n/q}}{r^{n-1}}\int_0^1 \frac{t^{n/q}}{t^n}\|Du\|_{L^p(B(x_0,r))}\,dt$$

$$= C\|Du\|_{L^p(B(x_0,r))}\, r^{1-n/p}\underbrace{\int_0^1 t^{-n/p}\,dt}_{=\frac{p}{p-n}}$$

So

$$|u(x)-u(y)| \leq |u(y)-u_{B(x_0,r)}| + |u_{B(x_0,r)} - u(x)|$$

$$\leq C(n,p)\, r^{1-n/p}\,\|Du\|_{L^p(B(x_0,r))} \qquad \forall x,y\in B(x_0,r)$$

Choose e.g. $x_0 = \frac{x+y}{2}$, $r = |x-y| \Rightarrow x,y\in B_{x,y} \equiv B(x_0,r) \Rightarrow$

$$\boxed{\begin{aligned} |u(x)-u(y)| &\leq C(n,p)\,|x-y|^{1-n/p}\,\|Du\|_{L^p(B_{x,y})} \\ &\leq C(n,p)\,|x-y|^{1-n/p}\,\|Du\|_{L^p(\mathbb{R}^n)}, \quad \forall u\in C^1\cap W^{1,p}(\mathbb{R}^n) \end{aligned}}$$

2° $u \in W^{1,p}(\mathbb{R}^n)$ general $\Rightarrow$ let $u_\varepsilon := \eta_\varepsilon * u$ be a standard mollification. Case 1° $\Rightarrow$

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C(n,p)\,|x-y|^{1-n/p}\,\|Du_\varepsilon\|_{L^p(\mathbb{R}^n)}$$

But Exercises 4 $\Rightarrow$ $u_\varepsilon(x) \to u(x)$ at Lebesgue points of u!

Since by standard L^p-theory $Du_\varepsilon = \eta_\varepsilon * Du \to Du$ in L^p-norm, $\|Du_\varepsilon\|_{L^p} \to \|Du\|_{L^p}$. Thus taking limit $\varepsilon \to 0$ get

$$|u(x) - u(y)| \leq C(n,p)\,|x-y|^{1-n/p}\,\|Du\|_{L^p(\mathbb{R}^n)}$$ at Lebesgue points x, y !

Since a.e. point is a Lebesgue point (Real analysis I) the claim follows. $\square$

Remark 1° If x is a Lebesgue point of u, let $x \in B(x_0,1) \Rightarrow$

$$|u(x)| \leq |u(x) - u_{B(x_0,1)}| + |u_{B(x_0,1)}|$$
$$\leq \frac{1}{|B(x_0,1)|}\int_{B(x_0,1)} |u(x)-u(y)|\,dy + \frac{1}{|B(x_0,1)|}\int_{B(x_0,1)} |u|\,dy$$
$$\underset{\text{Morrey + Hölder}}{\leq} C\|Du\|_{L^p(\mathbb{R}^n)} + C_n\|u\|_{L^p(\mathbb{R}^n)} = C(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Thus $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(n,p)\,\|u\|_{W^{1,p}(\mathbb{R}^n)}$, $n < p < \infty$.

2° By above, each $u \in W^{1,p}(\mathbb{R}^n)$ has a continuous repr. $\bar{u}$, with

$\gamma = 1 - n/p$

$$\|\bar u\|_{C^{0,\gamma}(\mathbb{R}^n)} := \|\bar u\|_{L^\infty} + \sup_{x,y\in\mathbb{R}^n} \frac{|\bar u(x) - \bar u(y)|}{|x-y|^\gamma} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

On compact embeddings

0.0. Remarks on embeddings of Banach spaces

Ex. Recall e.g. the consequence of Morrey's inequality:

(*) $\| u \|_{C^{0,\gamma}(\mathbb{R}^n)} \leq c(n,p) \, \| u \|_{W^{1,p}(\mathbb{R}^n)}$, $\forall \, u \in W^{1,p}(\mathbb{R}^n)$
$[\gamma = 1 - n/p]$

This says, besides that $W^{1,p} \subset C^{0,\gamma}$, when $n < p < \infty$, also little more; (*) also says, or can be interpreted as saying, that the identity operator

(**) $\quad Id : W^{1,p}(\mathbb{R}^n) \to C^{0,\gamma}(\mathbb{R}^n)$

is a continuous linear operator between these Banach spaces! This point of view is often useful, and we say that $W^{1,p}(\mathbb{R}^n)$ is <u>continuously</u> <u>embedded</u> to $C^{0,\gamma}(\mathbb{R}^n)$.

<u>Question</u> Can we say more of the operator (**)?

On compactness

Recall: A metric space $(\underline{X}, d)$ is precompact, if for all $\varepsilon > 0$, $\underline{X}$ can be covered by finitely many sets of diameter $\leq \varepsilon$,

$$\underline{X} = \bigcup_{j=1}^{n} A_j, \quad dia(A_j) \leq \varepsilon.$$

Also, $\underline{X}$ is compact if every open cover of $\underline{X}$ has a finite subcover.

A.1. Proposition. If $(\underline{X}, d)$ metric space, following are equivalent.

(i) $\underline{X}$ is compact

(ii) Every sequence $(x_n)_1^\infty \subset \underline{X}$ has a converging subsequence [i.e. $\underline{X}$ sequentially compact]

(iii) $\underline{X}$ is precompact and complete.

[For a proof, see e.g. Väisälä's book Topology I]

Note: A set $A \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ A closed and bounded.

This fact is not true in ∞-dimensional Banach spaces:

A.2. Example If $0 \le \eta \in C_c^\infty(\mathbb{R}^n)$ & $\operatorname{supp}(\eta) \subset B(0,1)$, set

$$\eta_\ell(x) = \eta(x - 4\ell x_0), \quad \ell = 0, 1, 2, \ldots$$
$$|x_0| = 1.$$

Then $\operatorname{supp}(\eta_\ell) \subset B(4\ell x_0, 1)$ so that

$$\operatorname{supp}(\eta_\ell) \cap \operatorname{supp}(\eta_j) = \phi \quad \text{for } \ell \neq j.$$

Thus $\|\eta_\ell - \eta_j\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \eta_\ell(x)^p + \eta_j(x)^p \, dx = 2M^p > 0$,

where $M = \|\eta\|_{L^p(\mathbb{R}^n)} > 0$.

Hence $\{\eta_\ell\}_{\ell=1}^\infty$ has no converging subsequences in $L^p(\mathbb{R}^n)$.

On the other hand, $\|\eta_\ell\|_{L^p} = \|\eta\|_{L^p} = M \quad \forall \ell \Rightarrow \{\eta_\ell\}_1^\infty \subset B_{L^p}(0, M)$ is bounded.

A very useful criterium for compactness in the space $C(X) = \{ f : (X, d) \to \mathbb{C} \text{ continuous \& bounded} \}$, equipped with norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$ is given by the Ascoli–Arzela theorem:

A.3. Theorem (Arcoli-Arzela). If (X, d) is a compact metric space and $H \subset C(X)$, then

H is relatively compact (i.e. $\overline{H}$ compact in $C(X)$)

$\Longleftrightarrow$

H equicontinuous & pointwise bounded

Here:

- H equicontinuous at $x_0 \in X$, if $\forall\, \varepsilon > 0$ can find $\delta > 0$ s.t.

$$d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \quad \forall\, f \in H.$$

- H equicontinuous, if it is equicontinuous at every point $x_0 \in X$

- H pointwise bounded if $\forall\, x \in X$, the set $\{f(x) : f \in H\} \subset \mathbb{C}$ is bounded

A.4. Example: $\left.\begin{array}{c}\text{Morrey} \\ n < p < \infty\end{array}\right\} \Rightarrow |u(x) - u(y)| \le C\,|x-y|^{1-n/p}\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$

Thus $H := \{ u \in W^{1,p}(\mathbb{R}^n) : \|u\|_{W^{1,p}(\mathbb{R}^n)} \le 1 \}$ is equicontinuous in $\mathbb{R}^n$!

We also saw on page 5c that $\|u\|_\infty \le C \|u\|_{W^{1,p}(\mathbb{R}^n)}$;
Thus H pointwise bounded in $\mathbb{R}^n$!

However, H is <u>not</u> relatively compact in $C(\mathbb{R}^n)$.

(The example A.2 / p. 6c works also in $C(\mathbb{R}^n)$,

$\forall\, l \neq j$, $\|z_l - z_j\|_\infty = \sup\limits_{x \in \mathbb{R}^n} |\eta(x - 4l x_0) - \eta(x - 4j x_0)| = \|\eta\|_\infty > 0$)

<u>But</u> if $B = B(0, R)$,

$$H\big|_{\overline{B}} := \{u\big|_{\overline{B}} : u \in H\} = \{u\big|_{\overline{B}} : \|u\|_{W^{1,p}(\mathbb{R}^n)} \leq 1\}$$

then by Ascoli-Arzela $H\big|_{\overline{B}}$ is relat. compact in $C(\overline{B})$.

<u>Remark</u> If $\Omega \subset \mathbb{R}^n$ bdd domain with C^1-boundary, we can use the extension operator $E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ and Example A.4. above to get:

<u>A.5, Corollary</u> If $\Omega \subset \mathbb{R}^n$ is bdd domain with C^1-boundary, and if $n < p < \infty$, then

$$H := \{u \in W^{1,p}(\Omega) : \|u\|_{W^{1,p}(\Omega)} \leq 1\}$$

is relatively compact in $C(\overline{\Omega})$.

In order to systemize the above ideas let us recall that:

If X, Y Banach spaces and $T: X \to Y$ linear operator,

- T continuous / bounded if $\|Tx\|_Y \leq c\|x\|_X \quad \forall x \in X$
- T compact if in addition $T(B_X) \subset Y$ precompact (i.e. $\overline{T(B_X)}$ compact in Y)

Note: Above $B_X = \{x \in X : \|x\| \leq 1\}$ denotes the closed unit ball of the Banach sp. X

A.6. Definition If X, Y Banach spaces with $X \subset Y$ (and norms $\|\cdot\|_X$, $\|\cdot\|_Y$) say that X is compactly embedded in Y, written $X \subset\subset Y$, if

$$\mathrm{Id} : (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$$

is a compact operator, i.e. $\Longleftrightarrow$

(a) $\|x\|_Y \leq C \|x\|_X \quad \forall x \in X$ and

(b) Each bounded sequence of X has a subsequence that converges in Y, i.e. in the norm $\|\cdot\|_Y$.

Note: (b) $\Leftrightarrow$ B_X rel. compact in Y: Indeed $A \subset X$ bounded $\Rightarrow$ $A \subset B_X(0, M) = M\, B_X(0,1)$ for some $M < \infty$; thus

$A = \mathrm{Id}(A) \subset M\, \mathrm{Id}(B_X)$ rel. compact when $X \subset\subset Y$.

A.7. Example. (If $n < p < \infty$ and) If $\Omega \subset \mathbb{R}^n$ bdd domain with C^1-boundary $\partial\Omega$, then $\boxed{W^{1,p}(\Omega) \subset\subset C(\bar{\Omega})}$, by Corollary A.5 / p. 6e and Remark 1° / p. 5c.

A Remark on compact embeddings of $W^{1,p}(\Omega)$

Thm Suppose $\Omega \subset \mathbb{R}^n$ bdd domain with C^1-bdry $\partial\Omega$.
Then

a) $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ if $1 \le p < n$ and $q < p^*$.

b) $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ $\forall\, 1 \le q < \infty$ if $n \le p < \infty$.

Proof: a) is Rellich-Kondrachov Thm; Evans Thm 1/5.7

b) Since Ω bounded, $W^{1,p}(\Omega) \subset W^{1,s}(\Omega)$ if $s < p$. (by Hölder)

Hence if $s < n \le p$ and $1 \le q < \infty$ is given, note that $s^* = \frac{ns}{n-s} \to \infty$ when $s \to n$; thus can choose $s < n$ with $q < s^*$.

In particular

$$W^{1,p}(\Omega) \underset{\text{cont. embedding}}{\subset} W^{1,s}(\Omega) \underset{\text{Rellich-Kondrachov}}{\subset\subset} L^q(\Omega) \qquad \square$$

On weak convergence and weak compactness.

(i) If E Banach space, let $B_E = \{x \in E : \|x\| \leq 1\}$ be the closed unit ball of E.

(ii) The dual

$$E^* = \{x^* : E \to \mathbb{C} \text{ linear and continuous}\} \text{ with}$$

$$\|x^*\| = \sup\{|\langle x^*, x\rangle| : \|x\| \leq 1\}$$

For our purposes, can think $E = L^p(\Omega)$, $1 \leq p < \infty$, so that (can isometrically identify) $E^* = L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.
In this case,

$$\langle g, f\rangle = \int_\Omega f(x)\, g(x)\, dx \qquad \forall f \in L^p(\Omega), \; \forall g \in [L^p(\Omega)]^* = L^q(\Omega).$$

(iii) A sequence $(x_n)_1^\infty \subset E$ converges weakly to $x \in E$, write $x_n \xrightarrow{w} x$, if

$$\lim_{n \to \infty} \langle x^*, x_n\rangle = \langle x^*, x\rangle \qquad \forall x^* \in E^*.$$

In case of $L^p(\Omega)$, $f_n \xrightarrow{w} f$ if

$$\int_\Omega f_n(x)\, g(x)\, dx \xrightarrow[(n \to \infty)]{} \int_\Omega f(x)\, g(x)\, dx \quad \forall g \in L^q(\Omega)$$

iv) Weak topology $\equiv$ topology induced by the family $\{x^* \in E^*\}$, see [Väisälä, Topologia II].

Then

$$x_n \xrightarrow{w} x \iff x_n \text{ converges in the weak topology to } x.$$

Note: If E^* separable $\left[\exists \{x_n^* : n \in \mathbb{N}\} \subset E^* \text{ dense, i.e. } \overline{\{x_n^*\}} = E^*\right]$

then on B_E the weak topology is metrizable!

Metric: $$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\langle x_n^*, x-y\rangle|}{1+|\langle x_n^*, x-y\rangle|}$$

In particular, this holds in $L^p(\Omega)$!

v) Reflexive Banach spaces:

- $\forall x \in E$, $x^* \mapsto \langle x^*, x\rangle$ continuous & linear.

  That is, $E \subset (E^*)^* = E^{**}$ ($\equiv$ bidual of E)

- E reflexive $\underset{\text{def.}}{\iff}$ $E = E^{**}$

Example: If $1 < p < \infty$, then $L^p(\Omega)^* = L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$

$\Rightarrow$ $L^p(\Omega) = L^q(\Omega)^* = L^p(\Omega)^{**}$ is reflexive!

[But $L^1(\Omega)$, $L^\infty(\Omega)$ are not reflexive]

vi) <u>Alaoglu's theorem</u>

If E reflexive, then B_E compact in the weak topology.

[Remark: The general version of Alaoglu's thm. considers dual Banach spaces with so called w^*-topology]

vii) <u>Corollary</u>: If E is separable and reflexive, then B_E is

- compact in the weak topology.
- metrizable —"— —"— —"—
- sequentially compact —"— —"— —"—

<u>Thus</u>: If $\{x_n\}_1^\infty \subset E$ bounded, can find $x \in E$ s.t.

$$x_{n_\ell} \xrightarrow{w} x \quad \text{for a subsequence } \{x_{n_\ell}\}.$$

viii) <u>Important Consequence</u>: If $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$, let $\{f_n\}_1^\infty \in L^p(\Omega)$ be <u>any</u> sequence with $\|f_n\|_{L^p(\Omega)} \leq C_0 < \infty \quad \forall n$.

Then can find $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ so that $f_{n_k} \xrightarrow{w} f$, i.e.

$$\int_\Omega f_{n_k}(x)\, g(x) \xrightarrow[(k\to\infty)]{} \int_\Omega f(x) g(x)\, dx \quad \forall g \in L^q(\Omega).$$

A Lemma on weak limits in $L^p(\Omega)$

Lemma Suppose $\|f_k\|_{L^\infty(\Omega)} \leq M_0 < \infty$, $k = 1, 2, 3, \dots$, and $|\Omega| < \infty$.

If $f_k \xrightarrow{\omega} f$ in $L^p(\Omega)$, for some $1 \leq p < \infty$, then $\|f\|_{L^\infty(\Omega)} \leq M_0$.

Note: Norm bounds remain under weak limits!

Proof of Lemma If $g \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_\Omega f_k(x)\, g(x)\, dx \xrightarrow[(k\to\infty)]{} \int_\Omega f(x)\, g(x)\, dx$$

In particular,

$$(*) \quad \left| \int_\Omega f(x)\, g(x)\, dx \right| \leq \lim_{k\to\infty} \int_\Omega \|f_k\|_\infty |g(x)| dx \leq M_0 \int_\Omega |g(x)| dx$$

Let then $E_\varepsilon := \{ x \in \Omega : |f(x)| \geq (1+\varepsilon) M_0 \}$ $(\varepsilon > 0)$

If $|E_\varepsilon| > 0$, set $g(x) = \frac{\overline{f(x)}}{|f(x)|} \chi_{E_\varepsilon}(x)$. Then

$$M_0 \int_\Omega |g(x)| dx = M_0 |E_\varepsilon| \quad , \text{ but}$$

$$\left| \int_\Omega f(x)\, g(x)\, dx \right| = \left| \int_{E_\varepsilon} |f(x)|\, dx \right| \geq (1+\varepsilon) M_0 |E_\varepsilon|$$

And this contradicts $(*)$!! Thus $|E_\varepsilon| = 0$ and $\|f\|_\infty \leq M_0$ $\square$

Question Can you generalize above to $\|f_k\|_{L^s(\Omega)} \leq M_0$, $s \geq p$?

Finding weak solutions via Riesz representation thm

We look for weak solutions $u \in W_0^{1,2}(\Omega)$ to

$$(A) \quad Lu := -\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j} u\right) + c(x)\,u = f \quad \text{in } \Omega$$

i.e.

$$-\nabla \cdot a(x)\nabla u + c(x)\,u = f, \qquad x \in \Omega,$$

where $a(x)$ are uniformly elliptic symmetric matrices, i.e. $\exists\ 0 < \lambda \leq \Lambda < \infty$ s.t.

$$\lambda |\xi|^2 \leq \sum_{i,j} \xi_i\, a_{ij}(x)\, \xi_j \leq \Lambda |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n,$$

and $c(x) \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

Remark: Requirement $u \in W_0^{1,2}(\Omega)$ means we are looking for weak solutions to the bdry value problem

$$\begin{cases} Lu = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

Riesz representation theorem: If H Hilbert space with inner product $(\cdot,\cdot)$ and if $F: H \to \mathbb{C}$ is linear and continuous, then there is a unique $w \in H$ such that

$$F(h) = (h, w), \qquad \forall h \in H.$$

We also need a form of Poincaré inequality

B. Lemma If $\Omega \subset \mathbb{R}^n$ bdd domain and $u \in W_0^{1,2}(\Omega)$, then

$$\int_\Omega u^2\,dx \le c(n)\,\mathrm{diam}(\Omega)^2 \int_\Omega |Du|^2\,dx$$

Proof: We know from lectures & Exercises 5 that $W_0^{1,2}(\Omega) \subset L^2(\Omega)$, i.e. $\|u\|_{L^2(\Omega)} \le c\,\|u\|_{W_0^{1,2}(\Omega)}$.

To estimate c, note that we have some $c_0 < \infty$ s.t.

$$\int_{B(0,1)} u^2 \le c_0 \int_{B(0,1)} |Du|^2\,dx \quad, u \in W_0^{1,2}(B(0,1))$$

By scaling, i.e. considering $u(Rx)$, we get

$$\int_{B(0,R)} u^2 \le c_0 R^2 \int_{B(0,R)} |Du|^2\,dx \quad, u \in W_0^{1,2}(B(0,R))$$

By translating this holds in every ball $B(x_0,R)$, since the zero-extension of $u \in W_0^{1,2}(\Omega)$ is in $W_0^{1,2}(B(x_0,R))$ when $\Omega \subset B(x_0,R)$, the claim follows from choosing $R = \mathrm{diam}(\Omega)$. $\square$

To use these results to solving (A) define a new inner product in $W_0^{1,2}(\Omega)$,

$$(C) \quad \langle u,v\rangle = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}(x) D_i u\, D_j v + c(x)\, u v \Big)\,dx$$

That $\langle u,v\rangle$ is indeed an inner product follows from

D. Lemma There is a constant $c_0 < 0$ such that if $c \geq c_0$, then $\langle \cdot , \cdot \rangle$ is a positive definite innerproduct on $W_0^{1,2}(\Omega)$.

Proof: write $|||u|||^2 = \langle u, u \rangle$. Here

$$\langle u, u \rangle = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}(x) D_i u \, D_j u + c u^2 \Big) dx$$

ellipticity

$$\geq \lambda \int_\Omega |Du|^2 dx + c_0 \int_\Omega u^2 dx$$

Poincaré, Lemma B

$$\geq \frac{\lambda}{2} \int_\Omega |Du|^2 + \Big(\frac{\lambda}{2 c(n) \operatorname{dia}(\Omega)^2} + c_0 \Big) \int_\Omega u^2 dx$$

$$\geq \alpha \, \| u \|^2_{W_0^{1,2}(\Omega)}$$

Here $\alpha = \min \left\{ \frac{\lambda}{2}, \frac{\lambda}{2 c(n) \operatorname{dia}(\Omega)^2} + c_0 \right\} > 0$ if

(E) $$c_0 > - \frac{\lambda}{2 c(n) \operatorname{dia}(\Omega)^2} = - \frac{c(n, \lambda)}{\operatorname{diam}(\Omega)^2} .$$

Thus $\langle u, u \rangle = 0 \Rightarrow u = 0$; the other properties of an inner product are clear. $\square$

Remark. As $|||u|||^2 = \langle u, u \rangle \leq \Lambda \int_\Omega |Du|^2 + \| c \|_\infty \int_\Omega u^2 dx$

$\leq \max \{ \Lambda, \| c \|_\infty \} \, \| u \|^2_{W_0^{1,2}(\Omega)}$, the norms $|||\cdot|||$ and $\| \cdot \|_{W_0^{1,2}(\Omega)}$ are equivalent!

F. Theorem. There exists a constant $c_0 < 0$ such that the PDE

$$(\ast) \qquad -\sum_{i,j=1}^{n} D_i\left(a_{ij}(x) D_j u\right) + c(x)\, u = f$$

has a unique weak solution $u \in W_0^{1,2}(\Omega)$ for every $f \in L^2(\Omega)$ provided $c(x) \geq c_0$

Remark $\quad c_0 = - \dfrac{c(n)\lambda}{\operatorname{dia}(\Omega)^2}$ for some $c(n) > 0$.

Proof of Theorem: Let $\widehat{W_0^{1,2}(\Omega)}$ be $W_0^{1,2}(\Omega)$ equipped with the new equivalent inner product $\langle \cdot,\cdot \rangle$.
Then given $f \in L^2(\Omega)$,

$$F(v) := \int_\Omega v(x)\, f(x)\, dx$$

is a continuous linear map $F : \widehat{W_0^{1,2}(\Omega)} \to \mathbb{R}$;

$$|F(v)| \underset{\text{Hölder}}{\leq} \left(\int_\Omega f^2\, dx\right)^{1/2} \left(\int_\Omega v^2 dx\right)^{1/2} \leq \|f\|_{L^2(\Omega)} \|v\|_{W_0^{1,2}(\Omega)}$$

$$\leq c\, \|f\|_{L^2(\Omega)} |||v|||.$$

Hence, by Riesz representation, there is a unique $u \in \widehat{W_0^{1,2}(\Omega)}$ such that

$$\langle h, u \rangle = F(h) \qquad , \quad h \in \widehat{W_0^{1,2}(\Omega)};$$

i.e.

$$\int_\Omega \left(\nabla h \cdot a(x) \nabla u + c(x)\, h(x)\, u(x)\right) dx = \int_\Omega h(x)\, f(x)\, dx$$

In particular, this holds $\forall\, h = \varphi \in C_c^\infty(\Omega)$! Thus u is weak solution to $(\ast)$, and uniqueness follows from Riesz repr. $\square$

On Dirichlet's Principle

As an introductory example to Calculus of Variations consider first the linear PDE

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x) \partial_j u \right) + c(x) u = f$$

where $a_{ij}(x)$ unif. elliptic and $c \in L^\infty(\Omega)$ & $f \in L^2$ as before.

This PDE is associated to the variational integral

$$I(v) = \frac{1}{2} \int_\Omega \left(\sum_{i,j} a_{ij}(x) \partial_i v \, \partial_j v + c(x) v^2 \right) dx - \int_\Omega f(x) v \, dx$$

Example: For Poisson eqn $-\Delta u = f$ the associated variational integral is

$$I(v) = \frac{1}{2} \int_\Omega |Dv|^2 dx - \int_\Omega f(x) v \, dx$$

Definition. Given $g \in W^{1,2}(\Omega)$ let

$$\mathcal{A} = \mathcal{A}(g) := \{ u \in W^{1,2}(\Omega) : u - g \in W_0^{1,2}(\Omega) \}$$
$$= \{ u \in W^{1,2}(\Omega) : u|_{\partial\Omega} = g \text{ in the Sobolev sense} \},$$

and call $\mathcal{A}$ the set of admissible functions.

Remark: If Ω bdd with C^1-bdry $\partial\Omega$, then

$$\mathcal{A} = \mathcal{A}(g) = \{ u \in W^{1,2}(\Omega) : Tr(u) = g \}$$ Tr = Trace

Definition. A function $u \in W^{1,2}(\Omega)$ is a minimizer for the variational integral $I(u)$, with respect to the family $\mathcal{A}$, if

$$I(u) \leq I(v) \quad \forall v \in \mathcal{A}$$
$$(\text{and } u \in \mathcal{A}).$$

Remark: If $\mathcal{A} = \mathcal{A}(g) = \{u \in W^{1,2}(\Omega) : u - g \in W_0^{1,2}(\Omega)\}$ then a minimizer u minimizes $I(u)$ among all functions with the (same) boundary values g!

Theorem If $u \in W^{1,2}(\Omega)$ is a minimizer, w.r.t. $\mathcal{A} = \mathcal{A}(g)$, for the variational integral

$$(A) \qquad I(v) = \frac{1}{2}\int_\Omega \left[\nabla v \cdot a(x)\nabla v + c(x)v^2\right] dx - \int_\Omega f(x) v \, dx$$

then u is a weak solution to the PDE

$$(B) \qquad \begin{cases} -\nabla \cdot a(x)\nabla u + c(x)u = f \\ u|_{\partial\Omega} = g \end{cases}$$

Proof: Let $\varphi \in C_c^\infty(\Omega)$ and $\varepsilon \in \mathbb{R}$. Then the variations of u, $u + \varepsilon\varphi \in g + W_0^{1,2}(\Omega) = \mathcal{A}(g)$.

Thus

$$I(u) \leq I(u + \varepsilon\varphi) = i(\varepsilon), \text{ where}$$

$$i(\varepsilon) = \frac{1}{2}\int_\Omega [\nabla(u+\varepsilon\varphi) \cdot a(x)\nabla(u+\varepsilon\varphi) + c(x)(u+\varepsilon\varphi)^2 - \int_\Omega f(u+\varepsilon\varphi)\,dx$$

As $i(\varepsilon)$ has minimum at $\varepsilon = 0 \Rightarrow i'(0) = 0$ [if it exists] 11c

On the other hand; develope:

$$i(\varepsilon) = \frac{1}{2}\int_\Omega \left[\nabla u \cdot a(x)\nabla u + c(x)u^2\right]dx - \int_\Omega f u \, dx$$

$$+ \varepsilon \frac{1}{2}\int_\Omega \nabla u \cdot a(x)\nabla \varphi + \nabla\varphi \cdot a(x)\nabla u + \frac{\varepsilon}{2}\int_\Omega 2c(x)u\varphi \, dx - \varepsilon\int_\Omega f\varphi \, dx$$

$$+ \varepsilon^2 \frac{1}{2}\int_\Omega \left[\nabla\varphi \cdot a(x)\nabla\varphi + c(x)\varphi^2\right]dx.$$

Thus $i(\varepsilon)$ is smooth in ε, with

$$i'(0) = \int_\Omega \left[\nabla\varphi \cdot a(x)\nabla u + c(x)\varphi u\right]dx - \int_\Omega f\varphi \, dx$$

Since $i'(0) = 0 \ \forall \varphi \in C_c^\infty(\Omega)$ we see that u is a weak solution to the PDE in (B). Also $u\big|_{\partial\Omega} = g$ by assumption $\square$

<u>Remarks</u> 1° The PDE in (B) is called the Euler-Lagrange equation for the variational integral (A).

2° Does $I(v)$ have a minimizer for any given boundary values g ?!

3° More general variational integrals $\rightsquigarrow$

- Methods for their Euler-Lagrange equations!
- When do minimizers exist for general variational integrals ?!

Weierstrass' example of a variational integral not admitting a minimum.

In 1870 Weierstrass gave the following example:

Let $I(w) = \int_{-1}^{1} [x\, w'(x)]^2 \, dx$

Then there is no minimizer $u \in W^{1,2}(-1,1)$ (say) within the admissible functions $\mathcal{A} = \left\{ u \in W^{1,2}(-1,1),\ \begin{matrix} u(-1)=0 \\ u(+1)=2 \end{matrix} \right\}$

Namely let $w(x) = w_\varepsilon(x) = 1 + \dfrac{\arctan(x/\varepsilon)}{\arctan(1/\varepsilon)}$, $x \in [-1,1]$.

Then $w_\varepsilon'(x) = \dfrac{1}{\arctan(1/\varepsilon)} \; \dfrac{\varepsilon}{x^2+\varepsilon^2}$, with $w \in \mathcal{A}$,

so that

$$I(w_\varepsilon) \le \int_{-1}^{1} \frac{\cancel{x^2}}{\arctan^2(1/\varepsilon)} \; \frac{\varepsilon^2 \, dx}{(x^2+\varepsilon^2)^{\cancel{2}}}$$

$$\le \frac{\varepsilon}{[\arctan(1/\varepsilon)]^2} \int_{-1}^{1} \frac{\varepsilon \, dx}{x^2+\varepsilon^2} = \frac{\varepsilon}{[\arctan(1/\varepsilon)]^2} \int_{-1/\varepsilon}^{1/\varepsilon} \frac{dy}{y^2+1} \to 0$$

as $\varepsilon \to 0$

Thus $\inf \{ I(w) : w \in \mathcal{A} \} = 0$

But if $u \in W^{1,2}(-1,1)$ and $I(u) = 0 \Rightarrow u' = 0$ a.e.

So that u constant, $u(-1) = u(+1)$; Thus $u \notin \mathcal{A}$!

and $I(u)$ admits no minimizer in $\mathcal{A}$.

$\square$

Further results from Functional Analysis and weak convergence

Recall that if $\underline{X}$ Banach space and $\underline{X}^*$ its dual, then $|\langle x^*, x\rangle| \leq \|x\| \, \|x^*\| \quad \forall x \in \underline{X}, \ \forall x^* \in \underline{X}^*$,

With Hahn-Banach theorem you find:

(1) $$\|x\| = \sup\{ |\langle x^*, x\rangle| : x^* \in \underline{X}^*, \|x^*\| = 1 \}$$

(see FA-notes on webpage, Satz 9.19 p. 173)

Another consequence of the Hahn-Banach theorem is

Proposition I If M is a closed subspace of a Banach space $\underline{X}$ and $x_0 \in \underline{X}$ s.t.

$$d := \operatorname{dist}(x_0, M) > 0$$

then there is a $x^* \in \underline{X}^*$ such that

$$x^*\big|_M \equiv 0 \quad , \quad \langle x^*, x_0\rangle = d \ \ \& \ \ \|x^*\| = 1 .$$

(see FA-notes, Lemma 9.18, p. 172)

With these results we first show that $I(x) := \|x\|$ is weakly lower semicontinuous on $\underline{X}$.

Lemma FA1 Let $\underline{X}$ be a Banach space and suppose $x_k \xrightarrow{w} x$, i.e. sequence $\{x_k\}$ converges weakly to $x \in \underline{X}$.

Then
$$\|x\| \le \liminf_{k\to\infty} \|x_k\|$$

Proof: We use (1) from previous page; if $x^* \in \underline{X}^*$ and $\|x^*\|=1$, by weak convergence

$$|\langle x^*, x\rangle| = \lim_{k\to\infty} |\langle x^*, x_k\rangle| \overset{(1)}{\le} \liminf_{k\to\infty} \|x_k\|\,\|x^*\|$$

and using (1) again gives $\|x\| = \sup_{\|x^*\|=1} |\langle x^*, x\rangle| \le \liminf_{k\to\infty} \|x_k\|$. $\square$

Further results from FA:

Lemma FA2 If $M \subset \underline{X}$ is a closed subspace of a Banach space $\underline{X}$, if $x_k \in M$ $\forall k$ and if $x_k \xrightarrow{w} x$ in $\underline{X}$, then $x \in M$. [i.e. closed linear subspaces are closed under weak limits]

Proof: If $x \notin M$, then $d := \operatorname{dist}(x, M) \equiv \inf\{\|x-z\| : z \in M\} > 0$.

By Proposition I / previous page, there is $x^* \in \underline{X}^*$ so that

$$x^*|_M \equiv 0, \quad \langle x^*, x\rangle = d \quad \text{and} \quad \|x^*\| = 1. \quad \text{But then}$$

$$0 = \lim_{k\to\infty} \langle x^*, x_k\rangle \underset{\text{weak conv.}}{=} \langle x^*, x\rangle = d,$$ a contradiction! Thus $x \in M$. $\square$

A third result from basic FA we needed was

Lemma FA3 If $x_k \xrightarrow{w} x$ in a Banach space $\overline{X}$, then $\{x_k\}_{k=1}^{\infty}$ is bounded, i.e. $\sup_k \|x_k\| \le C < \infty$.

Proof: This is a consequence of Banach-Steinhaus theorem (FA-notes, Prop. 7.4, p. 135) Namely associate to each x_k a linear operator

$$T_k : \overline{X}^* \longrightarrow \mathbb{R}, \quad T_k(x^*) = \langle x^*, x\rangle .$$

Then $\| T_k \| = \sup\{ |T_k(x^*)| : \|x^*\| = 1\} = \sup\{ |\langle x^*, x_k\rangle| : \|x^*\| = 1\}$

$= \|x_k\|$ By (1), page 13a.

Now Banach-Steinhaus theorem says that either

$\sup_k \|T_k\| < \infty$ or there is $x^* \in \overline{X}^*$ for which $\sup_k |\langle x^*, x_k\rangle| = \infty$.

Since $\langle x^*, x_k\rangle \to \langle x^*, x\rangle$ $\forall x^* \in \overline{X}^*$, the latter cannot hold;

thus $\sup_k \|x_k\| = \sup_k \|T_k\| \le C < \infty$. $\square$

Return to Dirichlet Principle

In 11b / Theorem we showed that if the variational int.

$$I_D(v) := \frac{1}{2}\int_\Omega \left[\nabla v \cdot a(x)\nabla v + c(x)v^2\right]dx - \int_\Omega f(x)v(x)\,dx$$

has a minimizer u in $\mathcal{A}(g) = \{v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega)\}$,

then u is a weak solution to $\begin{cases} -\nabla\cdot a(x)\nabla u + c(x)u = f \\ u|_{\partial\Omega} = g \end{cases}$

The natural assumptions here are, besides $g \in W^{1,2}(\Omega)$, that

(1) $\lambda|\xi|^2 \le \xi\cdot a(x)\xi \le \Lambda|\xi|^2$, $c \in L^\infty$ and $f \in L^2(\Omega)$,

and $I_D(v)$ can be written as

(2) $I_D(v) = \int_\Omega L(Dv, v, x)\,dx$ where

(3) $L(p, z, x) = \frac{1}{2}p\cdot a(x)p + \frac{1}{2}c(x)z^2 - f(x)z$

From [Evans, §8.2/Theorem 2; p. 448] we know that if L is convex in p and coercive, $L(p,z,x) \ge \alpha|p|^q - \beta$ for some constant $\alpha > 0$, $\beta \ge 0$ then a minimizer exists.

However assuming only $f \in L^2$, the L in (3) is not bdd below, hence not coercive! [As in section 10 in the end one needs to assume $C(x) \geq c_0$ anyway, so the only real "problem" comes from the term "$f(x) z$"]

The case $f \in L^2(\Omega)$ requires thus a separate consideration which we provide here. The main lemma is the following:
(For simplicity consider the case $C(x) \equiv 0$)

Lemma D1. If $\mathcal{A}(g)$ is as on the previous page and $v \in \mathcal{A}(g)$, then

$$\int_\Omega v(x)^2 dx + \int_\Omega |\nabla v|^2 dx \leq c_1 I_D(v) + c_2$$

Proof: Note that by unif. ellipticity,

$$(4) \quad \int_\Omega |\nabla v|^2 dx \leq \underbrace{\frac{2}{\lambda} \frac{1}{2} \int_\Omega \nabla v \cdot a(x) \nabla v + \frac{2}{\lambda} \int_\Omega f(x) v(x)}_{= \frac{2}{\lambda} I_D(v)} - \frac{2}{\lambda} \int_\Omega f(x) v(x) dx.$$

Here

$$(5) \quad \int_\Omega |f(x)| |v(x)| dx \leq \frac{1}{\varepsilon} \int_\Omega |f|^2 dx + \varepsilon \int_\Omega |v|^2 dx$$

Moreover, $\|v\|_{L^2(\Omega)} \leq \|v - g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}$,
where $v - g \in W_0^{1,2}(\Omega)$ by assumption; thus we can use Poincare's inequality [Evans, Theorem 3/page 265] to get

$$\|v - g\|_{L^2(\Omega)} \leq c \|Dv - Dg\|_{L^2(\Omega)}$$
$$\leq c \|Dv\|_{L^2} + c \|Dg\|_{L^2}$$

Collecting the info we have $\|v\|_{L^2(\Omega)} \leq c \|Dv\|_{L^2(\Omega)} + c_1(g) \Rightarrow$

by squaring have

$$(6) \quad \int_\Omega v(x)^2\,dx \le 4C^2 \int_\Omega |\nabla v|^2\,dx + C_2(g)$$

and using this in (4) & (5) have

$$\int_\Omega |\nabla v|^2\,dx \le \frac{2}{\lambda} I_D(v) + \frac{2}{\lambda}\left|\int_\Omega f(x)v(x)\,dx\right|$$

$$\le \frac{2}{\lambda} I_D(v) + \frac{2}{\lambda}\varepsilon\, 4C^2 \int_\Omega |\nabla v|^2\,dx + \varepsilon \frac{2}{\lambda} C_2(g) + \frac{1}{\varepsilon}\int_\Omega |f|^2\,dx$$

Let us choose here ε so small that $\varepsilon \frac{2}{\lambda} 4C^2 < \frac{1}{2}$ $\Rightarrow$

$$\frac{1}{2}\int_\Omega |\nabla v|^2\,dx \le \frac{2}{\lambda} I_D(v) + C_3 \quad , \quad C_3 = \varepsilon\frac{2}{\lambda}C_2(g) + \frac{1}{\varepsilon}\int_\Omega |f|^2\,dx$$

Finally, combining this with (6) proves the claim. $\square$

With the a'priori bound of Lemma D1, which replaces the coercivity assumptions, we can proceed quickly. But first weak lower semicontinuity:

<u>Lemma D2.</u> If $u_k \xrightarrow{w} u$ in $W^{1,2}(\Omega)$, then

$$I_D(u) \le \liminf_{k\to\infty} I_D(u_k)$$

<u>Proof:</u> Since

$$0 \le (\nabla u_k - \nabla u)\cdot a(x)(\nabla u_k - \nabla u) =$$

$$\nabla u_k \cdot a \nabla u_k + \nabla u \cdot a \nabla u - 2\nabla u_k \cdot a \nabla u \quad ,$$

we have

$$I_D(u_k) = \frac{1}{2}\int_\Omega \nabla u_k \cdot a(x)\nabla u_k - \int_\Omega f(x)\, u_k\, dx$$

$$\geq -\frac{1}{2}\int_\Omega \nabla u \cdot a(x)\nabla u + \underbrace{\int_\Omega \nabla u_k \cdot a(x)\nabla u}_{\xrightarrow{(k\to\infty)} \int_\Omega \nabla u \cdot a(x)\nabla u} - \underbrace{\int_\Omega f(x)\, u_k\, dx}_{\xrightarrow{(k\to\infty)} \int_\Omega f(x)\, u(x)\, dx}$$

[The limits by weak convergence; $|a(x)\nabla u| \in L^2$ (ellipticity) & $f \in L^2$ (assumption)]

Thus $\liminf_{k\to\infty} I_D(u_k) \geq -\frac{1}{2}\int_\Omega \nabla u \cdot a(x)\nabla u + \int_\Omega \nabla u \cdot a\nabla u - \int_\Omega f(x)\, u\, dx$

$$= +\frac{1}{2}\int_\Omega \nabla u \cdot a(x)\nabla u\, dx - \int_\Omega f(x) u(x)\, dx = I_D(u). \quad \square$$

Theorem D3 $\forall\, g \in W^{1,2}(\Omega)$ the variational int $I_D(u)$ has a minimum in $\mathcal{A}(g)$.

Proof: Note first that $\inf_{\mathcal{A}(g)} I_D(u) < \infty$, since

$$I_D(g) \leq \Lambda \int_\Omega |\nabla g|^2 + \int_\Omega f(x) g(x) \leq C\|Dg\|^2_{L^2} + \|f\|_{L^2}\|g\|_{L^2} < \infty.$$

If u_k minimizing sequence, by Lemma D1 $\{u_k\}$ bounded in $W^{1,2}(\Omega)$; thus it has a subsequence converging weakly,

$u_{k_\ell} \xrightarrow{w} u$, and Lemma D2 $\Rightarrow$ $I_D(u) \leq \liminf_{k\to\infty} I_D(u_k) = \inf_{v\in\mathcal{A}(g)} I_D(v)$

It remains to show $u \in \mathcal{A}(g)$. But $u_k - g \in W_0^{1,2}(\Omega)$ and

$u_k - g \xrightarrow{w} u - g$, so that 13b/Lemma FA2 shows that $u - g \in W_0^{1,2}(\Omega) \Rightarrow u \in \mathcal{A}(g)$. $\square$

In particular, combining Theorem D3 and 11b/Theorem solves the Dirichlet problem

Corollary D4. If $g \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$, there is a weak solution $u \in W^{1,2}(\Omega)$ to

$$\begin{cases} -\nabla \cdot a(x) \nabla u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & (\text{i.e. } u - g \in W_0^{1,2}(\Omega)) \end{cases}$$

Remark. Since e.g. Poincaré inequality was used in Lemma D.1 only for $W_0^{1,2}(\Omega)$ functions, no smoothness on $\partial\Omega$ was required! i.e. Corollary D4 works in all bounded domains $\Omega \subset \mathbb{R}^n$.