Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 9

Solutions to the exercises are to be returned by **Thursday May 19** to Petri Ola, office D329.

Recall that a smooth $L(P,z,x)$ is a *null Lagrangian*, if either of the following two equivalent conditions hold,

a. Every $f \in C^\infty(\Omega;\mathbb{R}^m)$ satisfies the Euler-Lagrange equations

$$-\nabla_x \cdot D_{P^k} L(Df,f,x) + D_{z^k} L(Df,f,x) = 0 \qquad \text{in } \Omega, \quad k = 1,\dots,m.$$

b.
$$\int_\Omega L(Df,f,x)dx = \int_\Omega L(Dg,g,x)dx,$$

whenever $f,g \in C^\infty(\overline{\Omega};\mathbb{R}^m)$ with $f = g$ on $\partial\Omega$.

1. Consider maps $f = (u,v,w) \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with differential matrix

$$Df(x) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

Show that the 2×2 minor $L(Df) := \det \begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} = v_y w_z - v_z w_y$ is a null-Lagrangian.

2. [Evans, Problem 8.7.7] Prove that $L(P) := trace(P^2) - trace(P)^2$ is a null Lagrangian. Here the trace of an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n$ is defined by $trace(A) = \sum_{j=1}^n a_{jj}$.

3. [Evans, Problem 8.7.4] Assume $\eta : \mathbb{R}^n \to \mathbb{R}$ is C^1.

(i) Show that $L(P,z,x) := \eta(z)\det P$ is a null Lagrangian; here $P \in \mathbb{M}^{n\times n}$, $z \in \mathbb{R}^n$.

(ii) Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, then

$$\int_\Omega \eta(f)\det(Df)dx$$

depends only on $f_{|\partial\Omega}$.

4. [Evans, Problem 8.7.5] If $f : \mathbb{R}^n \to \mathbb{R}^n$ is as in Problem 3, fix $x_0 \notin f(\partial\Omega)$. If r is so small that $B(x_0, r) \cap f(\partial\Omega) = \emptyset$, choose a C^1-map η so that $\int_{\mathbb{R}^n} \eta(z)dz = 1$ and $\eta(x) = 0$ when $|x - x_0| \geq r$.

Define

$$deg(f, x_0) = \int_\Omega \eta(f) \det(Df)dx,$$

the *degree* of f relative to x_0. Prove that the degree is an integer.

5. In geometric function theory one studies the *distortion* of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Writing $f = (u, v)$ and assuming that the Jacobian $\det(Df(x)) > 0$ is positive almost everywhere, the distortion is defined by

$$K(f) := \frac{|\partial_x u|^2 + |\partial_y u|^2 + |\partial_x v|^2 + |\partial_y v|^2}{\det(Df)}$$

Show that the functional $L(Df) := K(f)$ is polyconvex; do this by first showing that $F(x, y) = x^2/y$ is convex on $(0, \infty) \times (0, \infty)$.

[Hint: You need to show that $F(x, y) - F(a, b) \geq 2ab^{-1}(x - a) - ab^{-2}(y - b)$]

Note. In higher dimensions the distortion of a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$K(f) := \frac{\left[\sum_{j,k=1}^n |\partial_{x_j} f^k|^2\right]^{n/2}}{\det(Df)}$$

so that $K(tf) = K(f)$ for all $t \in \mathbb{R}$. Also in higher dimensions the distortion is polyconvex, but the algebra to prove this is a little more difficult.