Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 8

Solutions to the exercises are to be returned by **Thursday April 28** to Petri Ola, office D329.

1. Suppose $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary. Assume also that $A(x) = (a_{i,j}(x))$ is symmetric and uniformly elliptic, so that $\lambda|\xi|^2 \le \xi \cdot A(x)\xi \le \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$.

We know from the lectures that the variational integral

$$I(u) = \int_\Omega Du(x) \cdot A(x)Du(x) + f(x)u(x)\,dx$$

has a minimizer in the set $\mathcal{A}(g) := \{v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega)\}$. Prove that $I(\frac{1}{2}(u+v)) < \frac{1}{2}I(u) + \frac{1}{2}I(v)$ for $u, v \in W^{1,2}(\Omega)$ unless $u = v$ almost everywhere, and use this to show that the minimiser is unique.

2. [Evans, Problem 8.7.8] Explain why the methods studied in the lectures, i.e. Evans Chapter 8.2, will *not* work for the integral representing the area of the graph of a function,

$$I(w) = \int_\Omega \left(1 + |Dw|^2\right)^{1/2} dx,$$

over $\mathcal{A}(g) = \{w \in W^{1,2}(\Omega) : w - g \in W_0^{1,2}(\Omega)\}$ for any $1 \le q < \infty$.

3. Given $g \in W^{1,2}(\Omega)$ show that the Dirichlet problem

$$\begin{cases} -\Delta u + u^3 = 0, \\ u - g \in W_0^{1,2}(\Omega) \end{cases}$$

has at least one weak solution $u \in W^{1,2}(\Omega)$, if $\Omega \subset \mathbb{R}^4$ is bounded with C^1-boundary.

[Hint: Express u as a solution to the Euler-Lagrange equation of a suitable variational integral.]

4. If $a, b \in \mathbb{R}$ and $0 < t < 1$, define $w : \mathbb{R} \to \mathbb{R}$ by

$$w(s) = \begin{cases} as, & \text{if} \quad 0 \le s < t, \\ bs + t(a-b), & \text{if} \quad t \le s \le 1, \\ w(s-n) + nw(1), & \text{if} \quad n < s \le n+1, \quad n \in \mathbb{N}. \end{cases}$$

Given $0 \neq x_0 \in \mathbb{R}^n$ let then $u_k(x) = w(kx \cdot x_0)/k$.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, show that the sequence $u_k(x) := w(kx) \in W^{1,q}(\Omega)$, for every $1 \le q \le \infty$ and $k \in \mathbb{N}$. Furthermore, show that for $1 < q < \infty$ the sequence $\{u_k\}_{k \in \mathbb{N}}$ converges weakly in $W^{1,q}(\Omega)$, and *determine* its weak limit $u \in W^{1,q}(\Omega)$.

[Hint: Draw the graph of $w(s)$ and recall Problem 2 in Exercises 7]

5. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is *not* convex, so that there are $z_0, y_0 \in \mathbb{R}^n$ and $0 < t < 1$ so that $F(tz_0 + (1-t)y_0) > tF(z_0) + (1-t)F(y_0)$ and assume that F is bounded. Let $\Omega = B(0,1)$ be the unit ball of $\mathbb{R}^n$.

Show that the variational integral

$$I(u) = \int_\Omega F(Du)\, dx$$

is *not* weakly lower semicontinuous in any $W^{1,q}(\Omega)$, $1 < q < \infty$.

[Hint: Consider first the case $tz_0 + (1-t)y_0 = 0$; use here Problem 4]