Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 7

Solutions to the exercises are to be returned by **Thursday April 14** to Petri Ola, office D329.

1. a) Derive the Euler-Lagrange equations for the variational integral

$$I(u) = \int_\Omega F\big(Du(x)\big)dx,$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

b) Consider the variational integral $I(u) = \int_{-1}^{1}[x\,u'(x)]^2 dx$ from the counterexample of Weierstrass [c.f. the notes/Section 12, on homepage]. Find the solutions to the Euler-Lagrange equation of this variational integral.

[Hint: Non-constant solutions will have a singularity at $x = 0$.]

2. [Evans 8.6.1.b] Consider weakly converging sequences $(u_k)_{k=1}^{\infty} \subset L^p(0,1)$, where $1 < p < \infty$; see notes/Section 8, on homepage.

If $a, b \in \mathbb{R}$ and $0 < \lambda < 1$, let

$$u_k(x) \;=\; \begin{cases} a, & \text{if} \quad j/k \le x < (j+\lambda)/k, \\ b, & \text{if} \quad (j+\lambda)/k \le x < (j+1)/k. \end{cases} \qquad (j = 0, \ldots, k-1)$$

[Draw a picture] Show that $(u_k)_{k=1}^{\infty}$ converges weakly to $u(x) \equiv \lambda a + (1-\lambda)b$ in $L^p(0,1)$.

3. [Evans 8.6.2] Find $L = L(p, z, x)$ so that the PDE

$$-\Delta u + D\phi \cdot Du = f \qquad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to the functional $I(w) = \int_\Omega L\big(Dw, w, x\big)dx$. Here ϕ, f are given functions smooth in $\overline{\Omega}$.

4. [Evans 8.6.3] The *elliptic regularisation* of the heat equation is the PDE

$$(*) \qquad \partial_t u - \Delta u \,-\, \varepsilon\,\partial_t^2 u = 0 \qquad \text{in } \Omega_T,$$

where $\varepsilon > 0$, $\Omega_T = \Omega \times (0,T]$ and $\Omega \subset \mathbb{R}^n$. Show that (*) is the Euler-Lagrange equation corresponding to an energy integral

$$I_\varepsilon(w) = \int_{\Omega_T} L_\varepsilon\big(Dw, \partial_t w, w, x, t\big)\,dxdt.$$

[Here $Du = (\partial_{x_1}u, \ldots, \partial_{x_n}u)$ is the space gradient of u]

5. [Evans 6.6.2] A function $u \in W_0^{2,2}(\Omega) = H_0^2(\Omega)$ is a weak solution of the following boundary value problem for the *biharmonic equation*

$$(1) \qquad \begin{cases} \Delta^2 u \;=\; f, & \text{in } \Omega, \\ u \;=\; \frac{\partial u}{\partial \nu} \;=\; 0, & \text{on } \partial\Omega, \end{cases}$$

provided

$$\int_\Omega \Delta u \, \Delta v \, dx \;=\; \int_\Omega f\, v\, dx \qquad \forall v \in W_0^{2,2}(\Omega).$$

Given $f \in L^2(\Omega)$, prove that there always exists a weak solution to (1).