Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 6

Solutions to the exercises are to be returned by **Tuesday March 8** to Petri Ola, office D329.

1. Recall the continuous (i.e bounded) linear operators $T : X \to Y$ between Banach spaces X and Y, equipped with norm $\| T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$.

If $T_k : X \to Y$ are compact linear operators and $\|T - T_k\| \to 0$, show that $T : X \to Y$ is compact.
[Hint: Recall the different characterisations of compactness in Banach spaces]

2. Let $B = B(0, 1) \subset \mathbb{R}^2$. Then as discussed later, for $f \in L^p(B)$

$$u(x) := (Tf)(x) = \int_B \log |x - y| f(y) dy$$

is a solution to the Poisson equation $\Delta u = f$ in B. Show that for $2 < p < \infty$, $T : L^p(B) \to W^{1,p}(B)$ is a continuous linear operator. Deduce that T is compact as an operator $T : L^p(B) \to L^p(B)$.

[Hint: One approach is to prove $\|Tf\|_{W^{1,p}(B)} \leq C\|f\|_{L^p(B)}$ first for $f \in C_c^\infty(B)$. Another approach is to use difference quotients and Young's inequality]

[Note: Claim true also for $1 \leq p \leq 2$, but requires some more "machinery"]

3. Suppose $u \in W^{1,p}(\Omega)$, for some $1 < p < \infty$. If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous with $f(0) = 0$, use difference quotients to show that $f \circ u \in W^{1,p}(\Omega)$.

This is a (strong !) generalisation of Problem 4/Exercises 2. As an application, show that the positive part $u^+ \in W^{1,p}(\Omega)$; here $u^+(x) = u(x)$ if $u(x) \geq 0$ and $u^+(x) = 0$ otherwise.

4. Suppose $1 < s \leq p < \infty$ and $|\Omega| < \infty$, so that $L^p(\Omega) \subset L^s(\Omega)$. If $\|f_k\|_{L^p(\Omega)} \leq 1$, $k = 1, 2, \ldots$ and if $f_k \to f$ weakly in $L^s(\Omega)$, show that

$$f \in L^p(\Omega) \qquad \text{and} \qquad \|f\|_{L^p(\Omega)} \leq 1.$$

[Hint: Recall the $L^p - L^q$ duality; c.f. proof of "Lemma on weak limits in $L^p(\Omega)$" in notes on course web-page]

5. (Evans, problem 5.10.11) Recall the difference quotients $D_j^h u(x)$ and the difference gradient $D^h u(x) = (D_1^h u(x), D_2^h u(x), \ldots, D_n^h u(x))$.

Prove that Theorem 3 in Evans/Section 5.8 does not hold at $p = 1$: That is, show by an example that if we have $\|D^h u\|_{L^1(\Omega')} \le C$ for all $\Omega' \subset\subset \Omega$ and for all $|h| \le \text{dist}(\Omega', \partial\Omega))/2$, it does not necessarily hold that $u \in W^{1,1}(\Omega)$.