Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 5

Solutions to the exercises are to be returned by Thursday Feb. 25.
to Petri Ola, office D329.

Let X and Y be Banach spaces, $B_X = \{x \in X : \|x\| \leq 1\}$ be the (closed) unit ball of X and $T : X \to Y$ be a linear map. Recall that T is *continuous* if $\| Tx\| \leq C\|x\|$, $x \in X$, for some constant C, and that T is *compact* if, in addition, $T(B_X)$ is relatively compact/precompact in Y, see notes on the home page for more information, including the Ascoli-Arzela theorem.

1. a) Show that if $S : X \to Y$ and $T : X \to Y$ are compact operators, then $S + T : X \to Y$ is a compact operator.

b) If $T : X \to Y$ is a compact operator, and $S : Z \to X$, $R : Y \to W$ are continuous operators, for some Banach spaces Z and W, show that
$TS : Z \to Y$ and $RT : X \to W$ are compact operators.

2. Suppose $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded subdomain.
a) Show that we have the compact embedding

$$W_0^{1,p}(\Omega) \subset\subset L^p(\Omega), \qquad 1 \leq p < \infty.$$

b) If $\phi \in C_c^\infty(\Omega)$ is given, show that

$$T : W^{1,p}(\Omega) \to L^p(\Omega), \qquad (Tu)(x) = \phi(x)u(x),$$

is a compact operator.

3. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous, and let $f_s(x) = f(sx)$ for $s, x \in [0, 1]$. Determine whether the set $H = \{f_s : 0 \leq s \leq 1\}$ is relatively compact in the space $C[0, 1] = \{g : [0, 1] \to \mathbb{R} \text{ continuous}\}$.

4. If $K : [0, 1] \times [0, 1] \to \mathbb{C}$ is continuous, and $T : C[0, 1] \to C[0, 1]$ is given by

$$(Tf)(x) = \int_0^1 K(x, t)f(t)dt,$$

show that T is a compact operator.

5. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary $\partial\Omega$. If

$$k < \frac{n}{p} \quad \text{and} \quad \frac{1}{q_0} = \frac{1}{p} - \frac{k}{n},$$

show that we have the compact embedding $W^{k,n}(\Omega) \subset\subset L^q(\Omega)$ for every $1 \leq q < q_0$.