Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 4

Solutions to the exercises are to be returned by Thursday Feb. 18.
to Petri Ola, office D329.

1. Suppose $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded subdomain. If $u \in W_0^{1,p}(\Omega)$, set

$$\overline{u}(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$$

Show that $\overline{u} \in W^{1,p}(\mathbb{R}^n)$.

2. Let $0 \leq \eta \in C_c^\infty(\mathbb{R}^n)$, with $\mathrm{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta(x)dx = 1$, be a standard mollifier and set $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$.

If $f \in L^1_{loc}(\mathbb{R}^n)$, show that we have

$$(\eta_\varepsilon * f)(x) \to f(x) \qquad \text{as } \varepsilon \to 0$$

at *every* Lebesgue point of f.

[Recall: x Lebesgue point of f if $\lim_{\varepsilon\to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} |f(y) - f(x)|dy = 0$.]

3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary $\partial\Omega$ and with trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$. If $\psi \in C^\infty(\overline{\Omega})$, show that

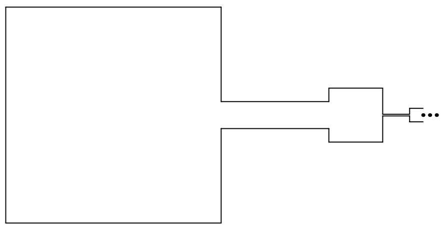
$$\psi T(u) = T(\psi u), \qquad \text{for all} \quad u \in W^{1,p}(\Omega).$$

4. If $u \in W^{1,p}(\Omega)$, show that then $|u| \in W^{1,p}(\Omega)$.

[Hint: Apply Problem 4 in Exercises 2, with the function $f(x) = f_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2} - \varepsilon$, and let $\varepsilon \to 0$.]

5. Show that there are bounded domains $\Omega \subset \mathbb{R}^2$ where the Gagliardo-Nirenberg-Sobolev inequality fails: At least for some $1 \leq p < n = 2$ and $p^* = \frac{2p}{2-p}$, there are functions $u \in W^{1,p}(\Omega) \setminus L^{p^*}(\Omega)$.

One possible class of such domains Ω, called "rooms and corridors", is described on the next page.



Rooms and corridors. Let $\Omega \subset \mathbb{R}^2$ be a domain such as in the picture above,

$$\Omega = \bigcup_{k=1}^{\infty} (D_k \cup P_k)\,,$$

where the 'fat' sets D_k, *the rooms*, and the 'thin' sets P_k, *the corridors*, $k = 0, 1, 2, \ldots$, are defined as follows:

Let first $d_k = 1 - 2^{-k}$, $k = 0, 1, 2, \ldots$, and define then the rooms as cubes

$$D_k = (d_{2k}, d_{2k+1}) \times \left(-2^{-2k-2}, 2^{-2k-2}\right)$$

and the corridors as rectangles

$$P_k = [d_{2k+1}, d_{2k+2}] \times \left(-\varepsilon_k 2^{-2k-2}, \varepsilon_k 2^{-2k-2}\right).$$

[Hint for solving Problem 5: Choose u to be constant c_k in each room D_k, and let it grow linearly in each corridor P_k. Choose the constants c_k so that $u \in L^p(\Omega) \setminus L^{p^*}(\Omega)$, and then the thinnesses ε_k suitably to have $u \in W^{1,p}(\Omega)$]