Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 3

Solutions to the exercises are to be returned by Tuesday Feb. 9.
to Petri Ola, office D329.

1. Show that $u(x) = \log\log\left(1 + \frac{1}{|x|}\right) \in W^{1,n}(B)$, where $B = B(0,1)$ is the unit ball in $\mathbb{R}^n$.

2. a) Show that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$.

b) Prove the generalised Hölder's inequality: If $1 \leq p_1, \dots p_m \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$, then

$$\int_\Omega |u_1 u_2 \cdots u_m| dx \leq \prod_{j=1}^m \left(\int_\Omega |u_j|^{p_j} dx \right)^{1/p_j}.$$

3. If $\Omega = (a,b) \subset \mathbb{R}$, how do you define the trace operator T on $W^{1,p}(\Omega)$? If $u \in W^{1,p}(\Omega)$, show that $T(u) = 0$ if and only if the absolutely continuous representative of u satisfies $u(a) = u(b) = 0$.

4. Let $0 \leq \eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier, with $\operatorname{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta(x)dx = 1$. Set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$, and let $e_n := (0, \dots, 0, 1) \in \mathbb{R}^n$.

If $u \in W^{1,p}(\mathbb{R}^n_+)$ and $u_\varepsilon(x) = u(x + 2\varepsilon e_n)$ show that $w_\varepsilon := \eta_\varepsilon * u_\varepsilon$ is well defined in $\mathbb{R}^n_+$, and

$$w_\varepsilon \in C^\infty(\overline{\mathbb{R}^n_+}) \quad \text{with} \quad \|u - w_\varepsilon\|_{W^{1,p}(\mathbb{R}^n_+)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

5. Suppose $u \in W^{1,p}(\mathbb{R}^n_+)$ with weak derivatives $D^\alpha u \in L^p(\mathbb{R}^n_+)$, $|\alpha| = 1$, and let $\varphi \in C^1(\overline{\mathbb{R}^n_+})$.

Suppose $\varphi(x) = 0$ if $x \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ or if $|x| > M$. Show that

$$\int_{\mathbb{R}^n_+} u(x)\, D^\alpha \varphi(x)\, dx = -\int_{\mathbb{R}^n_+} D^\alpha u(x)\, \varphi(x)\, dx.$$

[Hint: Use e.g. ideas from proof of Theorem 2/Section 5.5/Evans]