Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 2

Solutions to the exercises are to be returned by Tuesday Feb. 2.
to Petri Ola, office D329.

1. If dimension $n = 2$, consider the domain $\Omega = B(0,1) \setminus [0,1]$, i.e. the disk minus a slit. Show that in this domain, $C^\infty(\overline{\Omega})$ is *not* dense in $W^{1,p}(\Omega)$.

2. Integrate by parts and approximate, to prove the interpolation inequality

$$\int_\Omega |Du|^2 dx \leq C \left(\int_\Omega u^2 dx \right)^{1/2} \left(\int_\Omega |D^2 u|^2 dx \right)^{1/2}$$

for all $u \in W_0^2(\Omega)$, the closure of $C_c^\infty(\Omega)$ in $W^2(\Omega)$.

3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, and assume that every point $x_0 \in \partial\Omega$ has a neighbourhood V such that $C^\infty(\overline{\Omega \cap V})$ is dense in $W^{1,p}(\Omega \cap V)$. Show that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.

4. Suppose $f \in C^1(\mathbb{R})$ with $f' \in L^\infty$ and $f(0) = 0$. If $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, show that $f \circ u \in W^{1,p}(\Omega)$ and we have the chain rule

$$D^\alpha (f \circ u)(x) = f'(u) D^\alpha u(x), \qquad |\alpha| = 1,$$

almost everywhere in Ω.
[Hint: Approximation can be useful also in this problem]

5. Suppose $\Phi : U \to V$ is a C^1-diffeomorphism between domains $U, V \subset \mathbb{R}^n$; in particular, $\Phi^{-1} : V \to U$ is also a C^1-smooth homeomorphism.
If $\Omega \subset\subset U$ and $\Omega' = \Phi(\Omega)$, show that

$$u \in W^{1,p}(\Omega') \Leftrightarrow u \circ \Phi \in W^{1,p}(\Omega),$$

with $c_1 \|u\|_{W^{1,p}(\Omega')} \leq \|u \circ \Phi\|_{W^{1,p}(\Omega)} \leq c_2 \|u\|_{W^{1,p}(\Omega')}$.