Department of Mathematics and Statistics
Sobolev Spaces
Exercise 1

Solutions to the exercises are to be returned by Tuesday Jan. 26.
to Petri Ola, office D329.

1. If $u(x) = |x|,\ x \in (-1,1)$, show that u does not have the weak second order derivative u'' in $\Omega = (-1,1)$.

[Recall that as discussed in the lectures, the weak first order derivative exists, $u'(x) = \mathrm{sign}(x)$.]

Mollifiers: Let $0 \le \eta \in C_c^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta(x)dx = 1$. (such exist by Real Analysis, or Evans, Appendix C.4.). Set

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\eta\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0.$$

2. If $u \in W^{k,p}(\Omega)$ and $\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$, set

$$u^\varepsilon(x) = (\eta_\varepsilon * u)(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y-x)u(y)dy, \qquad x \in \Omega_\varepsilon.$$

Show that $u^\varepsilon(x)$ is a well defined C^∞-function in Ω_ε and that

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \qquad \text{pointwise in } \Omega_\varepsilon.$$

3. If u_ε and Ω_ε are as in the previous problem, show that $u_\varepsilon \in W^{k,p}(\Omega_\varepsilon)$ and that $u^\varepsilon \to u$ in $W^{k,p}_{loc}(\Omega)$. That is: if $U \subset \Omega$ is an open subset with compact closure $\overline{U} \subset \Omega$, then $u^\varepsilon \to u$ in $W^{k,p}(U)$ as $\varepsilon \to 0$.

4. If $u \in W^{1,p}(\Omega)$ and the weak derivative $Du = 0$, show that $u(x) = C$ for a.e. x, for some constant C.

5. [Evans, problem 5.10.6] If $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, show that

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p\,dt\right)^{1/p} \qquad \text{for } a.e.\ x, y \in [0,1].$$

[Hint: Recall that in dimension $n = 1$ we have a characterisation of $W^{1,p}(0,1)$ in terms of absolutely continuous functions]