Schramm Loewner Evolution **Exercise session 1**
Spring 2016 **26.1.2016**

Exercise 1. [a] Suppose that $B = (B_t)_{t\geq 0}$ is a standard Brownian motion and define stochastic processes $W = (W_t^{(k)})_{t\geq 0}$, $k = 1,2,3$, by setting

$$W_t^{(1)} = B_{s+t} - B_s$$
$$W_t^{(2)} = \lambda^{-1/2} B_{\lambda t}$$
$$W_t^{(3)} = B_{T-t} - B_T$$

where $s \geq 0$, $\lambda > 0$, $T > 0$ are constants. Show that the stochastic processes $W = (W_t^{(k)})_{t\geq 0}$, $k = 1,2,3$, are standard Brownian motions.
Notice that $(W_t^{(3)})_{t\in[0,T]}$ is only defined up to time T (such a process is called BM stopped at time T), and you need to check the properties of BM only for the time interval $[0,T]$.

Exercise 2. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$. Denote the probability density of the random variable $B_t \sim \mathrm{N}(0,t)$ by $p_t\colon \mathbb{R} \to [0,\infty)$.
(a) Consider the function of two variables $(t,x) \mapsto p_t(x)$. Show that it satisfies the partial differential equation $\frac{\partial}{\partial t} p_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} p_t(x)$.
(b) Define the d-dimensional standard Brownian motion $(\underline{B}_t)_{t\in[0,\infty)}$ by setting

$$\underline{B}_t = B_t^{(1)} \underline{e}_1 + \cdots + B_t^{(d)} \underline{e}_d,$$

where the components $B^{(1)}, \ldots, B^{(d)}$ are independent standard Brownian motions on $\mathbb{R}$. Let $p_t\colon \mathbb{R}^d \to [0,\infty)$ be the density of the random vector $\underline{B}_t$. Show that

$$\frac{\partial}{\partial t} p_t(x_1,\ldots,x_d) = \frac{1}{2}\Delta p_t(x_1,\ldots,x_d), \quad \text{where } \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}. \qquad (\text{“heat equation”})$$

Exercise 3. (a) Calculate explicitly

$$\mathsf{E}\left[\exp\left(\lambda B_t\right)\right], \qquad \lambda \in \mathbb{R} \qquad (*)$$

(b) Develop $(*)$ into series in λ on both sides, change the order of summation and expectation and deduce explicit formulas for the moments $\mathsf{E}[B_t^n]$, $n = 1,2,3,\ldots$

††† **Exercise 4.** [b] Continue the setup of Exercise 1 and set

$$W_t^{(4)} = \begin{cases} tB_{1/t} & \text{, when } t > 0 \\ 0 & \text{, when } t = 0 \end{cases}.$$

Show that the stochastic process $W = (W_t^{(4)})_{t\geq 0}$ is a standard Brownian motion.

[a] See the course webpage for instructions on the exercise system.
[b] Bonus exercises are marked with † or †††, the latter ones are harder.

Turn page!

† **Exercise 5.** A Brownian motion $(W_t)_{t\geq 0}$ with an initial distribution $W_0 \sim \mu$, where μ is a probability measure on $\mathbb{R}$, can be constructed in the following way. Suppose that $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion and Z is an independent random variable with $Z \sim \mu$. Then set $W_t = Z + B_t$. Repeat Exercise 2 (a) for the stochastic process $(W_t)_{t\geq 0}$.

Notice. Correct answer uses carefully probability theory. Do not forget to justify the change of the order of integration and limit, for instance.