Exercises 1–3 are "compulsory" and 4–5 are "bonus problems" (marked always with † or †††).

Exercise 1. Show that any hull K can be approximated by a sequence of hulls K_n such that ∂K_n is piecewise analytic and

$$\lim_n f_{K_n}(z) = f_K(z) \text{ for any } z \qquad \text{and} \qquad \lim_n a_1(K_n) = a_1(K).$$

Show also that f_{K_n} extends continuously to $\overline{\mathbb{H}}$.

Exercise 2. Define $m(z) = -\overline{z}$ which is an injective antiholomorphic self-map of $\mathbb{H}$.

(a) Let $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be a locally growing family of hulls with a driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$. Find the driving term of $(m(K_t))_{t\in\mathbb{R}_{\geq 0}}$.

(b) Remember that SLE(κ), $\kappa > 0$, is a Loewner chain with a stochastic driving term $W_t = \sqrt{\kappa} B_t$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion. Show that SLE(κ), $\kappa > 0$, is *symmetric*, i.e., $(m(K_t))_{t\in\mathbb{R}_{\geq 0}}$ and $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ are equal in distribution. Is the random Loewner chain with the driving process $W_t = \sqrt{\kappa} B_t + \alpha t$ symmetric?

Exercise 3. Let[a] $\nu = \pm 1$, $\kappa > 0$ and $h_t(z)$ be the solution of the differential equation

$$\partial_t h_t(z) = \nu \frac{2}{h_t(z) - W_t}, \qquad h_0(z) = z$$

with $W_t = -\sqrt{\kappa} B_t$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion with respect to a σ-algebra $\mathcal{F}_t$. Fix $z_0 \in \mathbb{H}$ and set $X_t = \operatorname{Re} h_t(z_0) - W_t$, $Y_t = \operatorname{Im} h_t(z_0)$ and $Z_t = X_t + \mathrm{i} Y_t$.

(a) Find expressions for $\mathrm{d}X_t$ (Itô differential of a semimartingale), $\partial_t Y_t$ and $\mathrm{d}Z_t$ (Itô differential of a $\mathbb{C}$-valued semimartingale).

(b) Calculate $\mathrm{d}\log Z_t$. Show that if $\theta_t = \arg Z_t$ and $\rho_t = |Z_t|$, then

$$\mathrm{d}\theta_t = \frac{(\kappa - 4\nu)\sin(2\theta_t)}{2\rho_t^2}\mathrm{d}t + \frac{\sqrt{\kappa}\sin\theta_t}{\rho_t}\mathrm{d}B_t.$$

(c) Find all smooth $F : (0,\pi) \to \mathbb{R}$ such that $M_t = F(\theta_t)$ is a martingale.

Set now $\nu = +1$, that is, we consider SLE(κ). Suppose for simplicity that there exists a simple curve $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that $K_t = \gamma[0,t]$ for all t and that $|\gamma(t)| \to \infty$ as $t \to \infty$. Then $\mathbb{H} \setminus \gamma(\mathbb{R}_{\geq 0})$ consists of two connected non-empty components L and R on the left and on the right of γ, respectively. Fix $z_0 \in \mathbb{H}$ and let $\tau = \inf\{t \geq 0 : z_0 \in K_t\}$. Suppose that $\tau = \infty$ almost surely.

(d) Define a process $M_t = \mathsf{P}[z_0 \in L \,|\, \mathcal{F}_t]$. Show that $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ is a martingale and that it can be written in the form $M_t = F(\theta_t)$.

(e) Suppose that F in (d) is smooth and extends continuously to $[0,\pi]$. Find an expression for M_0. For which values of κ this is possible?

[a] The parameter ν is here only for future use. If you wish you can set $\nu = +1$ already at this point.

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† **Exercise 4. Schwarzian derivative**

Define the Schwarzian derivative of f at z as

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

for any function f which is locally conformal near z, that is, f is holomorphic in a neighborhood of z with $f'(z) \neq 0$.

(a) Show that S satisfies

$$(S(f \circ g))(z) = g'(z)^2 Sf(g(z)) + Sg(z)$$

for any functions f and g that are locally conformal near $g(z)$ and z, respectively.

(b) Show that S satisfies

$$(S(\phi \circ f \circ \psi))(z) = \psi'(z)^2 Sf(\psi(z))$$

for all Möbius maps ϕ and ψ.

(c) Let A be a hull with $0 \notin A$ and define $\Phi_A(z) = g_A(z) - g_A(0)$. Show that

$$a_1(\tilde{A}) = -\frac{1}{6} S\Phi_A(0)$$

where $a_1(\tilde{A})$ is the half-plane capacity of the hull $\tilde{A} = \{-z^{-1} : z \in A\}$. Deduce that $S\Phi_A(0) < 0$ unless Φ_A is a identity map.

† **Exercise 5.** Let g_t and W_t be as above and let $a_k(t)$ be the coefficients in the expansion

$$g_t(z) = z + \sum_{k=1}^{\infty} a_k(t) z^{-k}.$$

In the Loewner equation, expand $2/(g_t(z) - W_t)$ in powers of z near ∞ and find a (closed) system of differential equations satisfied by $a_k(t)$ for $k = 1, 2, \ldots, n$ when $n = 5$. Integrate those equations to give explicit formulas for $a_k(t)$ in terms of the driving term.