Exercises 1–3 are "compulsory" and 4–5 are "bonus problems" (marked always with † or †††).

Exercise 1. Suppose that g_t is the solution of the Loewner equation with a continuous driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ and the corresponding growing family of hulls is $(K_t)_{t\in\mathbb{R}_{\geq 0}}$. Find the driving terms of the following growing families of hulls.

(a) $(x_0 + K_t)_{t\in\mathbb{R}_{\geq 0}}$ where $x_0 \in \mathbb{R}$.

(b) $(\lambda K_{t/\lambda^2})_{t\in\mathbb{R}_{\geq 0}}$ where $\lambda > 0$.

Hint. Find the right hydrodynamically normalized maps and differentiate.

Exercise 2. Let P^z be the law of a complex Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ sent from z. Let K be a hull and let $\tau_K = \inf\{t \in \mathbb{R}_+ : B_t \in \mathbb{R} \cup K\}$. Show that for any $z \in \mathbb{H} \setminus K$, $\mathsf{P}^z(\tau_K < \infty) = 1$ and the half-plane capacity $a_1(K)$ can be written as

$$a_1(K) = \lim_{y\nearrow\infty} y\, \mathsf{E}^{iy}\left(\operatorname{Im} B_{\tau_K}\right). \tag{1}$$

Exercise 3. Suppose that g_t is the solution of the Loewner equation with a continuous driving term $(W_t)_{t\in\mathbb{R}_+}$.

(a) Show that the inverse maps $f_t = g_t^{-1}$ satisfy the equation

$$\partial_t f_t(z) = -f_t'(z)\frac{2}{z - W_t}, \qquad f_0(z) = z.$$

(b) Show that the maps in reverse time $h_t = g_{T-t}$, $t \in [0, T]$, satisfy the equation

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - W_{T-t}}, \qquad h_0(z) = g_T(z), \qquad h_T(z) = z.$$

What are the equations satisfied by $\tilde{h}_t = g_{T-t} \circ f_T$?

† **Exercise 4.** (a) Let $w_0 \in \mathbb{H}$. Show that $z \mapsto (z - w_0)/(z - \overline{w_0})$ is a conformal map from $\mathbb{H}$ onto $\mathbb{D}$.

(b) Let K be a hull and g_K the corresponding hydrodynamically normalized map. Let $z_0 \in \mathbb{H}$ and $\tilde{g}_K(z) = (g_K(z) - g_K(z_0))/(g_K(z) - \overline{g_K(z_0)})$. Show that there exist constants C_1 and C_2 independent of K and z_0 such that

$$C_1 \operatorname{dist}(z_0, K \cup \mathbb{R}) \leq |\tilde{g}_K'(z_0)|^{-1} \leq C_2 \operatorname{dist}(z_0, K \cup \mathbb{R}).$$

Hint. Use the Koebe 1/4 theorem.

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† **Exercise 5.** Suppose that g_t is the solution of the Loewner equation with a continuous driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ and $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ is the corresponding growing family of hulls.

(a) Let $y > 0$. Show by expanding g_t around ∞ that as $y \to \infty$

$$\frac{1}{\pi}\operatorname{Im}\log(g_t(\mathrm{i}y) - W_t) = \frac{1}{2} + \frac{W_t}{\pi y} + \mathcal{O}(y^{-2}). \tag{2}$$

Here the branch of log is chosen so that $\log(g_t(z) - W_t)$ for large $z > 0$ is real and then extending holomorphically to $\mathbb{H} \setminus K_t$.

(b) Suppose that $K_t = \gamma([0,t])$ where γ is a simple curve. Interpret the left-hand side of (2) as harmonic measure in $\mathbb{H} \setminus K_t$ and write a formula similar to (1) for W_t as a limit of a quantity involving the harmonic measure as y tends to ∞.