Exercises 1–3 are "compulsory" and 4–5 are "bonus problems" (marked always with † or †††).

Exercise 1. Let $R > 0$ be fixed and let $f(z) = z + R^2 z^{-1}$.

(a) By a direct calculation, verify that $f(z) \in \mathbb{R}$ if and only if that $z \in \mathbb{R} \setminus \{0\}$ or $|z| = R$.

(b) Show that $f(z) = f(w)$ if and only if $z = w$ or $zw = R^2$.

(c) Show that f is a conformal map from $\mathbb{H} \setminus \overline{B(0,R)}$ onto $\mathbb{H}$.

Exercise 2 (Poisson kernel in $\mathbb{H} \setminus \overline{B(0,R)}$)**.**

(a) Let $R > 0$ and let $H_R = \mathbb{H} \setminus \overline{B(0,R)}$. Show that any continuous bounded function $u : \overline{H_R} \to \mathbb{R}$ which is harmonic in H_R and satisfies $u(x) = 0$, $x \in \mathbb{R} \setminus (-R, R)$, can be represented as an integral

$$u(z) = \frac{2R}{\pi} \int_0^\pi \frac{\operatorname{Im}(z + R^2 z^{-1}) \sin\theta}{|z + R^2 z^{-1} - 2R\cos\theta|^2} u(Re^{i\theta})\, \mathrm{d}\theta$$

for any $z \in H_R$. *Hint.* Make a conformal transformation to $\mathbb{H}$ and use the Poisson kernel of $\mathbb{H}$.

(b) Show that

$$\lim_{y \nearrow \infty} y\, u(iy) = \frac{2R}{\pi} \int_0^\pi u(Re^{i\theta}) \sin\theta\, \mathrm{d}\theta$$

Exercise 3. Let $\alpha, \beta \in (0,1)$ and $x_2 < x_1$. Let $f(z) = (z - x_1)^\alpha (z - x_2)^\beta$.

(a) Show that $f : \mathbb{H} \to \mathbb{C}$ is a Schwarz–Christoffel mapping and consequently that f is conformal.

(b) When $\beta = 1 - \alpha$, show that f has a simple pole at ∞, that is, show that $f(z) = c_{-1} z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$ around $z = \infty$. Calculate c_{-1}, c_0 and c_1.

Hint. Notice first that $(1 + z)^\alpha$ is holomorphic in the unit disc. Expand it around $z = 0$.

Turn page!

† **Exercise 4.** Let $n \geq 3$ and $\underline{B}_t = (B_t^{(1)}, \dots, B_t^{(n)})$ be a standard n-dimensional Brownian motion sent from $\underline{x} \neq (0,0,\dots,0)$. For a vector $\underline{x} \in \mathbb{R}^n$, denote its Euclidian norm by $\|\underline{x}\|$.

(a) Show that $Y_t = \|\underline{B}_t\|^{2-n}$ is a local martingale. *Hint.* Use Exercise 4 of Problem sheet 6. Notice that almost surely $\underline{B}_t \neq 0$ everywhere.

(b) Find all those $p > 0$ such that $\mathsf{E}[Y_t^p] < \infty$.

(c) Show that $\lim_{t\to\infty} \mathsf{E}[Y_t] = 0$. Is $(Y_t)_{t\in\mathbb{R}_{\geq 0}}$ a martingale? *Hint.* Show first that $\mathsf{E}[Y_t; \|\underline{B}_t\| \leq R]$ tends to 0 as $t \to \infty$ for fixed $R > 0$.

† **Exercise 5** (The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the spherical metric)**.**
Consider the complex plane as a subset of $\mathbb{R}^3$ by associating $\mathbb{C}$ with the the subspace of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 = 0$. A standard construction of the extended complex plane is through the *stereographic projection*: each point P on the sphere $S = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - 1/2)^2 = 1/4\} \subset \mathbb{R}^3$ other than $N = (0,0,1) \in S$ is projected to the complex plane by the taking the line going through P and N and finding the unique intersection point z_P of this line and the complex plane. As the point P approaches N, $|z_P|$ goes to infinity. This defines a mapping from the sphere S to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(a) For each $z \in \mathbb{C} \subset \mathbb{R}^3$, find $v_z \in S$ of the form

$$v_z = \lambda_z\, z + (1 - \lambda_z) N$$

where $0 < \lambda_z \leq 1$. Show that the map $z \mapsto v_z$ defines a smooth map from $\mathbb{C}$ onto $S \setminus \{N\}$ and that $v_z \to N$ as $|z| \to \infty$.

(b) Show that the map $z \mapsto v_z$ is conformal in the sense that the vectors $\partial_x v_z$ and $\partial_y v_z$ are orthogonal and have the same length. Here the partial derivatives are with respect to the real and imaginary parts of z.

(c) Find the *spherical metric* $\rho : \hat{C} \times \hat{C} \to \mathbb{R}$ defined by

$$\rho(z,w) = |v_z - v_w|,\ z,w \in \mathbb{C}, \qquad \rho(z,\infty) = |v_z - N|,\ z \in \mathbb{C}, \qquad (\ \rho(\infty,\infty) = 0\ )$$

where $|\cdot|$ is the Euclidian norm in $\mathbb{R}^3$. Show that the spherical metric is invariant under any map ϕ_A (as in Exercise 3 Problem Sheet 5) where $A \in \mathbb{C}^{2\times 2}$ is unitary, that is, $AA^* = I$ where A^* is the adjoint (conjugate transpose) of A.

Hint. For each $z \in \mathbb{C}$, define

$$\tilde{z} = \begin{bmatrix} z \\ 1 \end{bmatrix} \in \mathbb{C}^2$$

and notice that

$$\widetilde{\phi_A(z)} = \frac{1}{a_{21}z + a_{22}} A\tilde{z}.$$

What are the norms of $\tilde{z}$ and $\tilde{z} - \tilde{w}$?