UH/Department of Mathematics and Statistics **Exercise session 6**
Schramm–Loewner Evolution, Spring 2016 **1.3.2016**

Exercises 1–3 are "compulsory" and 4 is a "bonus problem" (marked always with † or †††).

In Exercise 2 you are asked to solve a Dirichlet problem. Remember that the solution of a Dirichlet problem satisfies a given PDE (here the Laplace equation $\Delta u = 0$) and given boundary values $u|_{\partial U} = \phi$.

Exercise 1. (a) Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Give at least two distinct examples of continuous functions on $\overline{\mathbb{H}}$ that are non-trivial, harmonic in $\mathbb{H}$ and vanish identically on $\mathbb{R}$.

(b) Let for each $\theta \in \mathbb{R}$ and $z \in \mathbb{D}$

$$P_\theta(z) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}.$$

Calculate ΔP_θ to show that P_θ is harmonic. Here as usual, $z = x + iy$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

(c) Suppose that h_n is a sequence of harmonic functions on a domain $U \subset \mathbb{C}$ such that h_n converge uniformly on any compact subset of U. Show that $\lim_n h_n$ is harmonic. *Hint.* Use a Poisson kernel.

Exercise 2. (a) Find u_1 and u_2 such that u_k, $k = 1, 2$, solves the following Dirichlet problem: u_k is harmonic in $\mathbb{H}$ and the boundary values are $u_k|_{\mathbb{R}} = \phi_k$ where

$$\phi_1(x) = \begin{cases} \pi & \text{, when } x < 0 \\ 0 & \text{, when } x > 0 \end{cases} \qquad \text{and} \qquad \phi_2(x) = \begin{cases} c_0 & \text{, when } x < x_1 \\ c_1 & \text{, when } x \in (x_1, x_2) \\ c_2 & \text{, when } x > x_2 \end{cases}$$

(b) Write conformal maps ϕ_k from $\mathbb{H}$ onto U_k, $k = 1, 2$, where $U_1 = \mathbb{R} \times (0, \pi)$, $U_2 = \mathbb{R} \times (a, b)$ and ϕ_k's satisfy $\phi_1(0) = -\infty$ and $\phi_1(\infty) = +\infty$ and $\phi_2(x_1) = -\infty$ and $\phi_2(x_2) = +\infty$. Here $a < b$ and $x_1 < x_2$ are real numbers.

(c) Let $\phi_3(z) = 1/z$ and $\phi_4(z) = z^\alpha$, $\alpha \in (0, 2)$. Verify directly that ϕ_k, $k = 1, 2, 3, 4$, are conformal maps on $\mathbb{H}$. What are the ranges of ϕ_k, $k = 3, 4$? What is the image of U_2 under the map $z \mapsto e^z$? When is that map injective (one-to-one)?

Notice. The quantity z^α is defined as $e^{\alpha \log z}$ where we choose the branch of $\log z$ so that $\log x$ is real for $x > 0$ and extend continuously to $\mathbb{H}$.

Exercise 3. Let f be holomorphic and $Z_t = X_t + iY_t$ be a complex semimartingale, i.e. the real and imaginary parts X_t and Y_t are semimartingales.

(a) Show that if $\langle Y \rangle_t = 0$ for all t, then $\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t + \frac{1}{2}f''(Z_t)\mathrm{d}\langle X \rangle_t$.

(b) Show that if $\langle X \rangle_t = \langle Y \rangle_t$ and $\langle X, Y \rangle_t = 0$ for all t, then $\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t$.

(c) In the general case, find an expression for $\langle Z \rangle_t$ so that Itô's formula can be written as $\mathrm{d}f(Z_t) = f'(Z_t)\mathrm{d}Z_t + \frac{1}{2}f''(Z_t)\mathrm{d}\langle Z \rangle_t$.

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††† **Exercise 4** (Bessel process).

Let $\delta \in \mathbb{R}$, $x > 0$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. A *Bessel process with dimension* δ *started from* x is the solution $(X_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{\delta - 1}{2X_t}\mathrm{d}t, \qquad X_0 = x$$

The theorem about SDEs from the lecture notes can be applied to show that the solution exists and is unique at least up to the stopping time

$$\tau = \sup\left\{t \in \mathbb{R}_+ : \inf_{s\in[0,t]} X_s > 0\right\}.$$

Denote the law of $(X_t)_{t\in[0,\tau)}$ by P^x.

(a) Let $\lambda > 0$. Show that the process $Y_t = \lambda X_{t/\lambda^2}$ is a Bessel process of dimension δ started from λx.

(b) For any $y \geq 0$, let $\sigma_y = \inf\{t \in [0,\tau] : X_t = y\}$. Let $0 < \varepsilon < x < L$. Find a local martingale of the form $f(X_t)$ such that f is twice differentiable, $f(\varepsilon) = 1$ and $f(L) = 0$. Show that $M_t = f(X_{t\wedge\sigma_\varepsilon\wedge\sigma_L})$ is a bounded martingale.

(c) Consider $X_t - B_t$ and show that $\sigma_\varepsilon \wedge \sigma_L$ is almost surely finite. Find $\mathsf{P}^x(\sigma_\varepsilon < \sigma_L)$ by applying optional stopping theorem to M_t.

(d) Use (c) to show that $\tau < \infty$ with positive probability if and only if $\delta < 2$. Show also that in that case, $\mathsf{P}(\tau < \infty) = 1$ and that $\lim_{t\nearrow\tau} X_t = 0$ almost surely.

Hint. For the last claim, you might need the strong Markov property of diffusions: if τ is an almost surely finite stopping time, then $Y_t = X_{\tau+t}$ is the solution of the same SDE with the initial value $Y_0 = X_\tau$.