UH/Department of Mathematics and Statistics **Exercise session 5**
Schramm–Loewner Evolution, Spring 2016 **23.2.2016**

Exercises 1–3 are "compulsory" and 4–5 are "bonus problems" (marked always with † or †††).

In Exercises 1 and 4, you are supposed to solve stochastic differential equations (SDE). Read more about SDEs from the lecture notes. Quite often, the best method for solving SDEs is to write a good guess and to apply Itô's formula.

Exercise 1 (Orstein–Uhlenbeck process)**.** Let $\alpha, \sigma \in \mathbb{R}$ be positive and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion.

(a) Solve the stochastic differential equation

$$\mathrm{d}X_t = -\alpha X_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t, \qquad X_0 = x_0 \in \mathbb{R}$$

by trying a solution of the form $X_t = a(t)\big(x_0 + \int_0^t b(s)\,\mathrm{d}B_s\big)$ where a and b are smooth enough functions on $\mathbb{R}_+$. This guess is motivated by the fact that we expect a Gaussian solution.

(b) Suppose that $X_0 = Z$ is a square-integrable random variable which is independent of $(B_t)_{t\in\mathbb{R}_+}$. Find $\mathsf{E}[X_t]$ and $\mathsf{Var}[X_t]$.

Exercise 2. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. For $x \in \mathbb{R}$ define $\tau_x = \inf\{t \in \mathbb{R}_+ : B_t = x\}$.

(a) Let $a < 0 < b$. Show that $\mathsf{P}(\tau_a \wedge \tau_b < \infty) = 1$ by considering $\mathsf{P}(B_t < a \text{ or } B_t > b)$.

(b) Apply the optional stopping theorem to $\mathsf{E}(B_{\tau_a\wedge\tau_b})$ and find the probabilities of the events $\{B_{\tau_a\wedge\tau_b} = a\}$ and $\{B_{\tau_a\wedge\tau_b} = b\}$ for $a < 0 < b$.

(c) Show that for all $x \in \mathbb{R}$, $\tau_x < \infty$ almost surely.

Exercise 3. (a) For $A \in \mathbb{C}^{2\times 2}$ with $\det A \neq 0$, define a Möbius map by

$$\phi_A(z) = \frac{a_{11}\, z + a_{12}}{a_{21}\, z + a_{22}}$$

where $A_{ij} = a_{ij}$. Show that

$$\phi_A \circ \phi_B = \phi_{AB}$$

and that $\phi_A^{-1} = \phi_{A^{-1}}$. Write an explicit formula for the inverse $\phi_{A^{-1}}(z)$ in terms of the elements of A and simplify it.

(b) Show that a map $\phi : \mathbb{D} \to \mathbb{D}$ is conformal and onto if and only if

$$\phi(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. *Hint.* Use Schwarz lemma for the "only if" claim.

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† **Exercise 4.** Let $x_0, x_1 \in \mathbb{R}$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. Find the solution of

$$\mathrm{d}X_t = \frac{x_1 - X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t, \quad t \in [0,1), \qquad X_0 = x_0$$

by slightly adapting the guess solution of Exercise 1. Find the mean $\mathsf{E}(X_t)$ and the covariance $\mathsf{E}[(X_s - \mathsf{E}(X_s))(X_t - \mathsf{E}(X_t))]$ of this process.

† **Exercise 5** (Time change of a semimartingale)**.**
(a) Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let $a(t,\omega)$ be a continuous, positive, adepted process. Define a random time-change by setting:

$$S(t,\omega) = \int_0^t a(r,\omega)^2 \,\mathrm{d}r, \qquad \sigma(s,\omega) = \inf\{t \in \mathbb{R}_+ : S(t,\omega) \geq s\}$$

Let

$$\tilde{B}_s(\omega) = \int_0^{\sigma(s)} a(r,\omega)\,\mathrm{d}B_r(\omega),$$

By the theorem on time changes of local martingales in the lecture notes, $(\tilde{B}_s)_{s\in\mathbb{R}_+}$ is a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_{\sigma(s)})_{s\in\mathbb{R}_+}$ and therefore we know how to construct Itô integral with respect to $\tilde{B}_s$. Show that the following time-change formula holds: for continuous, adapted process $v(t,\omega)$

$$\int_0^s v(\sigma(q),\omega)\,\mathrm{d}\tilde{B}_q(\omega) = \int_0^{\sigma(s)} v(r,\omega)a(r,\omega)\,\mathrm{d}B_r(\omega)$$

Hint. Check this first for $v(r,\omega) = \mathbb{1}_{[\sigma(s_1),\sigma(s_2))}(r)$.

(b) Let X_t be a semimartingale

$$\mathrm{d}X_t(\omega) = u(t,\omega)\mathrm{d}t + v(t,\omega)\mathrm{d}B_t(\omega).$$

Show that the process $(\tilde{X}_s)_{s\in\mathbb{R}_+}$ defined by

$$\tilde{X}_s = X_{\sigma(s)}$$

is a semimartingale with respect to $(\mathcal{F}_{\sigma(s)})_{s\in\mathbb{R}_+}$ and $(\tilde{B}_s)_{s\in\mathbb{R}_+}$ and satisfies

$$\mathrm{d}\tilde{X}_s = \frac{u(\sigma(s))}{a(\sigma(s))^2}\mathrm{d}s + \frac{v(\sigma(s))}{a(\sigma(s))}\mathrm{d}\tilde{B}_s.$$