Exercises 1–3 are "compulsory" and 4 is a "bonus problem" (marked always with † or †††).

Exercise 1. We call a expression of the form

$$\mathrm{d}X_t = m_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t \tag{1}$$

the *Itô differential* of X_t. Let $r, q \in \mathbb{N}$. Find the Itô differentials of the following processes:

(a) $X_t = B_t^q$, (b) $X_t = (\sin B_t)^r$, (c) $X_t = B_t^q\,(\sin B_t)^r$.

Exercise 2. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a non-random square-integrable function. Show that

$$X_t = \int_0^t f(s)\,\mathrm{d}B_s$$

is normally distributed. Find the mean and the variance of X_t.

Hint. Do the exercise first for a simple f.

Exercise 3 (Integration by parts)**.**

(a) For for any semimartingales X_t and Y_t, show that the following integration by parts formula holds

$$\int_0^t X_s\,\mathrm{d}Y_s = X_t\,Y_t - X_0\,Y_0 - \int_0^t Y_s\,\mathrm{d}X_s - \langle X, Y\rangle_t.$$

Note. As implicitly defined in Itô's formula for semimartingales in the lecture notes, here the definition for the integrals $\int_0^t X_s\,\mathrm{d}Y_s$ and $\int_0^t Y_s\,\mathrm{d}X_s$ is such that

$$\int_0^t f_s\,\mathrm{d}X_s = \int_0^t f_s\,m_s\,\mathrm{d}s + \int_0^t f_s\,\sigma_s\,\mathrm{d}B_s \tag{2}$$

for the semimartingale of the form (1) whenever the two integrals on the right of (2) make sense. For more general semimartingales extend accordingly.

(b) Use Itô's formula for $B_t f(t)$ and $F(B_t)$, where f and F are smooth enough non-random functions, to find two integration by parts formulas for the integrals of types

$$\int_0^t f(s)\mathrm{d}B_s, \qquad \int_0^t f(B_s)\mathrm{d}B_s.$$

Note that this gives a way interpret these integrals in pathwise (ω-by-ω) sense. Why?

Turn page!

† **Exercise 4** (Modulus of continuity of Brownian motion)**.** (a) This exercise continues the exercises 2 and 3 of Problem sheet 3. Remember that there we made a construction of a Brownian motion on $[0,1]$ as a uniformly convergent series

$$B_t = \sum_{n=0}^{\infty} Z_t^{(n)}$$

where $(Z_t^{(n)})_{t\in[0,1]}$ are some piecewise linear processes. Remember also that for each $c > \sqrt{2\log 2}$ there exist an almost surely finite random variable N such that for $n \geq N$,

$$\|Z^{(n)}\|_\infty \leq c\, 2^{-\frac{n+1}{2}} \sqrt{n}.$$

Show that for any n and $h > 0$ and $0 \leq t \leq 1 - h$

$$\left|Z_{t+h}^{(n)} - Z_t^{(n)}\right| \leq \min\{h \cdot 2^n, 2\} \, \|Z^{(n)}\|_\infty$$

and use this bound to show that there exists a constant $C > 0$ and a random variable $\Delta > 0$ such that for any $0 < h < \Delta$,

$$|B_{t+h} - B_t| \leq C\sqrt{h \log(1/h)}.$$

Hint. For the last claim, divide the sum $\sum_{n=0}^{\infty} \left|Z_{t+h}^{(n)} - Z_t^{(n)}\right|$ as $\sum_{n=0}^{N-1} + \sum_{n=N}^{l} + \sum_{n=l+1}^{\infty}$ where N is as above. Then choose $\Delta > 0$ such that the first sum is less than $\sqrt{h\log(1/h)}$ and after that optimize over l.

(b) Show that for each $0 < \alpha < 1/2$ there exists an almost surely finite random variable $C_\alpha > 0$ such that for any $0 \leq s, t \leq 1$,

$$|B_t - B_s| \leq C_\alpha |t - s|^\alpha.$$